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Research article

Global well-posedness and viscosity vanishing limit of a new initial-boundary value problem on two/three-dimensional incompressible Navier-Stokes equations and/or Boussinesq equations

Shu Wang*

School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100124, People's Republic of China

* **Correspondence:** Email: wangshu@bjut.edu.cn.

Abstract: The global well-posedness theory and viscosity vanishing limit of the initial-boundary value problem on two/three-dimensional (2D/3D) incompressible Navier-Stokes (NS) equations and/or Boussinesq equations with nonlinear boundary conditions are studied. The global existence of weak solution to the initial boundary value problem for 2D/3D incompressible NS equation with one kind of boundary of pressure-velocity's relation and the global existence and uniqueness of the smooth solution to the corresponding problem in 2D case for large smooth initial data are proven. The viscosity vanishing limit of the corresponding initial-boundary value problem for 2D/3D incompressible NS equations in the bounded domain is also established. And the corresponding results are extended to the 2D/3D incompressible Boussinesq equations.

Keywords: global smooth solution; global existence of weak or smooth solution; incompressible NS equations and Boussinesq equations
Mathematics Subject Classification: 76D05, 76D03, 35Q30, 35Q35, 74F10

1. Introduction

We study the global well-posedness theory and viscosity vanishing limit of the following problem for two/three-dimensional (2D/3D) incompressible Navier-Stokes (NS) equations in the bounded domain

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = \varepsilon \Delta v + f_0, & x \in \Omega, t > 0, \\ \text{div } v = 0, & x \in \Omega, t > 0, \\ \varepsilon \dfrac{\partial v}{\partial n} - pn - \dfrac{1}{2}|v|^2 n = k_0 v + g_0, & x \in \partial\Omega, t > 0, \\ v(0,x) = v_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

and the following problem for 2D/3D incompressible Boussinesq equations in the bounded domain

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = \varepsilon \Delta v + \varrho e + f_0, & x \in \Omega, t > 0, \\ \text{div } v = 0, & x \in \Omega, t > 0, \\ \varrho_t + v \cdot \nabla \varrho = \kappa \Delta \varrho + f_1, & x \in \Omega, t > 0, \\ \varepsilon \dfrac{\partial v}{\partial n} - pn - \dfrac{1}{2}|v|^2 n = k_0 v + g_0, & x \in \partial\Omega, t > 0, \\ \kappa \dfrac{\partial \varrho}{\partial n} - \dfrac{1}{2}\varrho v \cdot n = k_1 \varrho + g_1, & x \in \partial\Omega, t > 0, \\ (v, \varrho)(0,x) = (v_0, \varrho_0)(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Here the functions $v = v(t) = v(t,x)$ and $p = p(t) = p(t,x)$ are the velocity vector and scalar pressure of the fluid respectively, and the function $\varrho = \varrho(t) = \varrho(t,x)$ is the temperature or density, Ω is the smooth bounded domain of $\mathcal{R}^d, d = 2,3$ with the boundary $\Gamma = \partial\Omega$, n is the unit outer normal vector to $\Gamma = \partial\Omega$, $x = (x', x_d) \in \Omega$ with $x' = x_1$ when $d = 2$ and $x' = (x_1, x_2)$ when $d = 3$, the vector e denotes the unit one of x_d-direct, the constants $\varepsilon > 0$ and $\kappa > 0$ are the viscosity and diffusion coefficients, and k_0, k_1 are fixed given constants. Also, the functions $f_0 = f_0(t) = f_0(t,x), f_1 = f_1(t) = f_1(t,x), g_0 = g_0(t) = g_0(t,x), g_1 = g_1(t) = g_1(t,x)$ are the given forces satisfying $\text{div } f_0(t) = 0$ and $v_0(x), \varrho_0(x)$ are the given initial data satisfying and $\text{div } v_0 = 0$.

It is well known that the 3D incompressible NS equations (with homogeneous boundary conditions in the case of the bounded domain) have at least one globally-in-time weak solution having the finite energy [1–4]. But, the issue of the global regularity and uniqueness for the weak solution in 3D case is still one open problem in the field of mathematical theory of fluid mechanics [5–11]. For global wellposedness theory on the smooth solution to some special cases for 3D incompressible NS equation, e.g., in the case of the axi-symmetric flow, see, for example, [6,8,12], in the case of the helical flow, see, for example, [13], and in the case of the absence of simple hyperbolic blow-up regimes for the three-dimensional incompressible Euler and quasi-geostrophic equations, see, for example, [14]. In the 2D case, global wellposedness and regularity theory of incompressible NS equation in the whole space and in the bounded domain with the classical boundary conditions, such as the Dirichlet boundary condition and Navier boundary condition are well-known, see [4,6,11]. Recently, in [15], the globally dynamical stabilizing effects of the geometry of the domain at which the flow locates and of the geometry structure of the solution to 3D incompressible NS equations are investigated, and the existence and uniqueness of the global and smooth solution to the Cauchy problem for 3D incompressible NS and Euler equations for a class of the large and smooth initial data in orthogonal curvilinear coordinate systems are also established.

Also, the two/three-dimensional incompressible Boussinesq equation is an important model in the applied sciences such as atmospheric and oceanographic, see [12, 16–18]. There are many important progresses on global wellposedness on incompressible Boussinesq equations. For example, for global smooth solution results of two-dimensional incompressible Boussinesq equations with the viscosity or diffusive coefficients in the smooth large initial data, see, for example, [8, 19–22], for global well-posedness in the 3D no-swirl case of axi-symmetric Boussinesq system with partial viscosity or partial thermal diffusivity, see [23–25]. Of course, global regularity theory for three-dimensional incompressible Boussinesq equation in the case of general smooth initial data is also still open.

Physically the boundary condition $(1.1)_3$ represents that the normal stress of the fluid at the boundary is a linear function of the velocity v if we use $p + \frac{1}{2}|v|^2$ as one new pressure function since we can re-write the equation $(1.1)_1$ into the form $\partial_t v + \text{curl } v \times v + \text{div }(-\epsilon\nabla v + (p + \frac{1}{2}|v|^2)I) = f_0$. Moreover, the boundary condition $(1.1)_3$ can be equivalently re-written into the following one

$$\begin{cases} (\varepsilon\dfrac{\partial v}{\partial n} - k_0 v - g_0) \times n = 0, & x \in \Gamma, t > 0, \\ p = \varepsilon\dfrac{\partial v}{\partial n} \cdot n - \dfrac{1}{2}|v|^2 - k_0 v \cdot n - g_0 \cdot n, & x \in \Gamma, t > 0. \end{cases}$$

Since, for the known study on wellposedness of initial-boundary value problem for incompressible NS/Euler or Boussinesq equations in one bounded domain with the boundary, the boundary conditions involved there are all given zero velocity Dirichlet/Navier boundary condition etc., so far we do not see any study progress on the problems (1.1) and (1.2) for incompressible fluid. In this paper we establish the global well-poseness theory and viscosity vanishing limit of a class of the initial-boundary value problem for the incompressible fluid with non-homogeneous unknown nonlinear boundary conditions of pressure-velocity relation's type or pressure-velocity coupled with density-velocity relation's type. Our main purpose is that we obtain the existence of the global weak solution to the problems (1.1) and (1.2) in 2D/3D cases and the global existence and uniqueness of the strong and smooth solution for the problems (1.1) and (1.2) in 2D case for any smooth and large initial data. Also, we want to establish the viscosity vanishing limit of the corresponding initial boundary value problem under the suitable assumption on the solution to the initial-boundary value problem for 2D/3D incompressible Euler equation with a class of nonlinear boundary condition.

Let us introduce basic difficulties and some key points of our success of this paper. Different from the traditional Cauchy problem and initial boundary value problem with homogeneous boundary condition for incompressible fluid, the main difficulty involved here is caused by the more complex nonlinear boundary condition $(1.1)_3$ or $(1.2)_{4,5}$, which yields to that the vorticity equation can not be used here since the boundary condition of the vorticity is very complex if one use the vorticity equation. More interesting, the boundary condition $(1.2)_{4,5}$ in the problem for incompressible Boussinesq equation shows that the non-homogeneous boundary condition $(1.2)_4$ for the velocity field shall transfer to the density to get the coupled non-homogeneous density-velocity relation's boundary condition $(1.2)_5$ due to the convection of the fluid. To deal with these complex boundary conditions, motivated by [4], where the global existence and uniqueness of the weak solution for incompressible NS fluid-structure interaction problem are established, we first introduce one new kind of definition of a class of global weak solution to the problem (1.1) or the problem (1.2), which is transformed into one equivalent integration system with nonlinear boundary integration term caused by the non-homogeneous nonlinear boundary condition $(1.1)_3$ or $(1.2)_{4,5}$. In this definition, the appearance of boundary integration term brings some difficulties when one obtain the more high regularities for the incompressible velocity of the involved approximating system by the Galerkin method. For example, we need to establish one new trace theorem (see Proposition 1, below, in this section) for the incompressible velocity field so as to obtain the uniform estimate of $\|\partial_t v^l(0)\|$ on $l = 1, 2, \cdots$, see (3.27) in section 3.1. Secondly, we construct the approximating solution by the Galerkin method and establish the uniformly a priori estimates for the approximating solution so as to obtain global weak solution by the compactness argument. In this steps, the key point of our success is to sufficiently use the advantage of the positive viscosity coefficient and the positive diffusion coefficient so as to perform the limit on nonlinear convection term and the boundary integration term in the present definition of global weak solution. Then, we obtain the more better regularity estimates in the boundary and also in the time direction so as to obtain the globally strong and smooth solution to the problem (1.1) and the problem (1.2) in 2D case for smooth and large initial data, where some important interpolation inequality and the trace inequality in two-dimensional case are used and the regularity theory for the boundary value problem for the Stokes equation with Dirichlet boundary condition can be used to conclude our desired results on the strong and smooth solution for smooth initial data satisfying suitable compatibility conditions.

We also point out that the nonlinear boundary condition $(1.1)_3$ or $(1.2)_{4,5}$ involved in this paper is in fact related to the free boundary value problem (see, for example, [26–31]) or the fluid structure interaction problems (see, for example, [4, 32–34]) for the incompressible fluid, where the boundary integration terms does not appear in the corresponding definition of the global weak solution because of Lagrangian transform or the cancellation between the fluid and the structure at the interface.

Now we recall some preliminary knowledge and prove one theorem for the trace of incompressible v velocity field, which will be used in our regularity estimates for the strong and smooth solution for incompressible fluid.

We recall some standard Hilbert and Sobolev spaces as follows ([6, 35–37]). The space $L^2(\Omega)$ is the Hilbert one with the inner product $(v, w)_\Omega$ and $L^2(\Omega)$ norm $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. The space $H^1(\Omega)$ and $H^s(\Omega)$ are the standard Sobolev one, and $H^s(\Gamma)$ is the standard fractional order Sobolev's space with $H^{-s}(\Gamma) = (H_0^s(\Gamma))'$ for $s > 0$. The spaces $W^{s,q}(\Omega)$ and $W^{s,q}(\Gamma) \subset L^q(\Gamma)$ are the Sobolev one. Denote $H = \{v \in L^2(\Omega) |\ \mathrm{div}\, v = 0\}$, $V_1 = \left\{w \in H^1(\Omega) |\ \mathrm{div}\, w = 0 \text{ on } \Omega\right\}$. Also, Y' denotes the dual space of the space Y, $v = (v_1, \cdot, v_d) \in Y$ denotes $v_i \in X, i = 1, \cdots, d$ or $v \in Y^d = Y \times \cdots \times Y$, $C > 0$ denotes a positive constant depending only upon the domain but independent of the given time T and $C(T) > 0$ denotes a positive constant depending upon T and the domain.

Some basic embedding results and basic trace inequalities are needed as follows [6, 36, 37].

Lemma 1. *If $\Omega \subset \mathcal{R}^d$ is one given smooth bounded domain satisfying $\Gamma = \partial\Omega \subset \mathcal{R}^{d-1}$, or $\Omega = \mathcal{R}^d_+ = \{x \in \mathcal{R}^d | y' = (y_1, \cdots, y_{d-1}) \in \mathcal{R}^{d-1}, y_d > 0\}$ is one half space with one boundary $\Gamma = \partial\Omega = \{x \in \mathcal{R}^d | y' = (y_1, \cdots, y_{d-1}) \in \mathcal{R}^{d-1}, y_d = 0\}$, then the embedding*

$$H^{s_1}(\Omega) \hookrightarrow L^{q_1}(\Omega), \tfrac{1}{q_1} = \tfrac{1}{2} - \tfrac{s_1}{d}, 0 \le s_1 < \tfrac{d}{2}, d = 1, 2, 3, \tag{1.3}$$

$$H^{s_2}(\Gamma) \hookrightarrow L^{q_2}(\Gamma), \tfrac{1}{q_2} = \tfrac{1}{2} - \tfrac{s_2}{d-1}, 0 \le s_2 < \tfrac{d-1}{2}, d = 2, 3 \tag{1.4}$$

and

$$H^s(\Omega) \hookrightarrow H^{s-\frac{1}{2}}(\Gamma), s > \frac{1}{2}, d = 2, 3 \tag{1.5}$$

are continuous. Moreover, there exists one linear continuous right inverse lifting mapping $R : w \in$

$H^{\frac{1}{2}}(\Gamma) \mapsto Rw \in H^1(\Omega)$ satisfying

$$Rw|_\Gamma = w, \qquad \|Rw\|_{H^1(\Omega)} \le C\|w\|_{H^{\frac{1}{2}}(\Gamma)} \tag{1.6}$$

for some positive constant C.

Also, the following trace inequality is basic:

$$\int_\Gamma |u|^2 d\Gamma \le \delta \int_\Omega |\nabla u|^2 dx + C\int_\Omega |u|^2 dx, \quad v \in H^1(\Omega) \tag{1.7}$$

for any $\delta > 0$.

We also recall one regularity result on the Stokes problem, see [6, 11, 38].

Lemma 2. *Assume that $\Omega \subset \mathcal{R}^d, d = 2, 3$, is one smooth bounded domain with the boundary $\Gamma = \partial\Omega \in C^s$, $s = \max\{l, 2\}$. Assume that $f_1 \in W^{l-2,q}(\Omega), g_1 \in W^{l-1,q}(\Omega), \Phi_1 \in W^{l-\frac{1}{q},q}(\Gamma)$, $l \ge 1, 1 < q < \infty$, and $\int_\Omega g_1 dx = \int_\Gamma \Phi_1 \cdot n d\Gamma$, where n is the unit outer normal vector on Γ. Then the solution (u, p) of the Stokes problem*

$$\begin{cases} \Delta u - \nabla p = f_1, & x \in \Omega, \\ div\, u = g_1, & x \in \Omega, \\ u = \Phi_1, & x \in \Gamma \end{cases} \tag{1.8}$$

satisfies the estimate

$$\|u\|_{W^{l,q}(\Omega)} + \|\nabla p\|_{W^{l-2,q}(\Omega)} \le C(\|f_1\|_{W^{l-2,q}(\Omega)} + \|g_1\|_{W^{l-1,q}(\Omega)} + \|\Phi_1\|_{W^{l-\frac{1}{q},q}(\Gamma)}). \tag{1.9}$$

To establish our regularity results on the strong and smooth solution for incompressible fluid, we state one theorem for the trace of incompressible velocity field $v \in H$, see, e.g., [39], and outline its proof for completeness, based on the lifting operator technique in Lemma 1. We point out that this result for general function $v \in L^2(\Omega)$ does not hold, i.e., the general trace operator $\mathcal{T}_0 : L^2(\Omega) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is not continuous, see [36].

Proposition 1. *The trace operator $\mathcal{T} : v \in H \mapsto (v \cdot n)|_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$ is continuous, i.e., there exists a positive constant $C = C(\Omega)$ such that, for $v \in H$, it holds*

$$\|(v \cdot n)|_\Gamma\|_{H^{-\frac{1}{2}}(\Gamma)} \le C\|v\|. \tag{1.10}$$

Proof. Here, for completeness, we outline the key points of the proof. For fixed $v \in H$ and for any $w \in H^{\frac{1}{2}}(\Gamma) = H_0^{\frac{1}{2}}(\Gamma)$, we define one functional

$$Z_v(w) = \int_\Omega v \cdot \nabla(Rw) dx, \tag{1.11}$$

where $R : H^{\frac{1}{2}}(\Gamma) \mapsto H^1(\Omega)$ is any linear and continuous right inverse operator satisfying (1.6).

We want to verify that, for any given $v \in H$, $Z_v(w)$ is one bounded linear functional defined on $H^{\frac{1}{2}}(\Gamma)$. Firstly, the operator $Z_v(w)$ is independent of the operator R. In fact, for any two operators R_1, R_2 satisfying the property (1.6), we have $(R_1 w - R_2 w)|_\Gamma = w - w = 0$ on Γ, and, hence

$$\int_\Omega v \cdot \nabla(R_1 w - R_2 w) dx = \int_\Gamma (v \cdot n)(R_1 w - R_2 w) d\Gamma - \int_\Omega \text{div}\, v(R_1 w - R_2 w) dx = 0 \tag{1.12}$$

due to div $v = 0$. Thus, we get from (1.12) that the functional $Z_v(w)$ is independent of the operator R and is determined uniquely by functions $v \in H$ and $w \in H^{\frac{1}{2}}(\Gamma)$.

Secondly, the operator $Z_v(w)$ is bounded on $H^{\frac{1}{2}}(\Gamma)$. In fact, by the definition (1.11), for any $w \in H^{\frac{1}{2}}(\Gamma)$,

$$\begin{aligned} |Z_v(w)| =& |\int_\Omega v \cdot \nabla(Rw)dx| \\ \leq& \|v\|\|\nabla(Rw)\| \\ \leq& \|v\|\|Rw\|_{H^1(\Omega)} \\ \leq& C\|v\|\|w\|_{H^{\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{1.13}$$

Also, it is obvious that $Z_v(w)$ is linear with respect to $w \in H^{\frac{1}{2}}(\Gamma)$.

Lastly, by Riesz-Frechet representation theorem, there exists $\gamma_0 v \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$Z_v(w) = (\gamma_0 v, w)_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}, \quad \|\gamma_0 v\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \|Z_v\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\|v\| \tag{1.14}$$

by using (1.13). On the other hand, for $v \in C^1(\overline{\Omega})$, div $v = 0$, and $w \in C^1(\overline{\Omega})$, we have

$$Z_v(w) = \int_\Omega v \cdot \nabla R(w|_\Gamma)dx = \int_\Omega v \cdot \nabla w dx = \int_\Gamma v \cdot n w d\Gamma = (v \cdot n, w)_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}. \tag{1.15}$$

Combining (1.14) and (1.15), we get $\gamma_0 v = (v \cdot n)|_\Gamma$ and (1.10).

This completes the proof of Proposition 1. □

Finally, we recall one result on the functional analysis ([40]).

Lemma 3. *Let the space Y be a separable Hilbert space, then the space Y has a complete orthogonal system consisting of an at most countable number of elements.*

The rest of this paper is organized as follows. In section 2 we state the main results of this paper, and section 3 gives the proofs of our main results.

2. Main Results

Introduce

$$a_\Omega(v, w) = (\nabla v, \nabla w)_\Omega, b_\Omega(v, w, z) = (v \cdot \nabla w, z)_\Omega.$$

2.1. On Incompressible Navier-Stokes equations

Definition 1. *(**The definition of the global weak solution to the problem** (1.1)) (v, p) is called to be one global weak solution to the problem (1.1) in time, if for any given positive T, there exists (v, p), defined in the interval $[0, T]$, satisfying $v \in L^\infty(0, T; L^2(\Omega)) \bigcap L^2(0, T; V_1)$, $\partial_t v \in L^s(0, T; V_1')$ for some constant $s > 1$,*

$$\begin{aligned} &(\partial_t v(t), w)_\Omega + \varepsilon a_\Omega(v(t), w) + b_\Omega(v(t), v(t), w) - \frac{1}{2}\int_\Gamma |v(t)|^2 n \cdot w d\Gamma \\ &\quad = \int_\Gamma (k_0 v(t) + g_0(t)) \cdot w d\Gamma + (f_0(t), w)_\Omega, w \in V_1, 0 \leq t \leq T, \\ &v(0) = v_0(x) \quad \textit{in } V_1'. \end{aligned} \tag{2.1}$$

Here $d\Gamma$ is the arc differential for $d = 2$ and the area differential for $d = 3$.

Remark 1. *We point out that if v is one global weak solution of the system (2.1) in the definition 1, then there exists a function p such that (v, p) satisfies the system (1.1) almost everywhere for (x, t) in some sense. In fact, setting $w \in C_0^\infty(\Omega)$ with div $w = 0$ in (2.1) and denoting*

$$\partial_t v - \epsilon\Delta v + v \cdot \nabla v - f_0 = S,$$

we have $S \in ((\mathcal{D}(\Omega \times (0, T)))')^d$ and $(S, w) = 0$ in the sense of $(\mathcal{D}((0, T)))'$ for any $w \in C_0^\infty(\Omega)$ with div $w = 0$, which gives $S = -\nabla p$ and the equation $(1.1)_1$ for some $p \in ((\mathcal{D}(\Omega\times(0, T)))')^d$. Also, $(1.1)_2$ is obvious because $v \in V_1$. Thus, the rest one is to verify the nonlinear boundary condition $(1.1)_3$. Multiplying $(1.1)_1$ by $w \in V_1$ and using integration by parts, we get

$$\begin{aligned}(\partial_t v, w)_\Omega + \varepsilon a_\Omega(v, w) + b_\Omega(v, v, w) + \int_\Gamma (pn - \varepsilon\frac{\partial v}{\partial n}) \cdot w d\Gamma \\ = (f_0, w)_\Omega, w \in V_1, 0 \le t \le T,\end{aligned}$$

which, together with (2.1), gives

$$\int_\Gamma (\varepsilon\frac{\partial v}{\partial n} - pn - \frac{1}{2}|v|^2 n - (k_0 v + g_0)) \cdot w d\Gamma = 0, v \in V_1, 0 \le t \le T. \tag{2.2}$$

Then $(1.1)_3$ is obtained from (2.2) and $\int_\Gamma n \cdot w d\Gamma = 0$ because $w \in V_1$ is arbitrary.

Now we state the main results of this paper on the problem (1.1) for 2D/3D incompressible NS equations.

Theorem 1. *Assume that the domain $\Omega \subset \mathcal{R}^d, d = 2, 3$, is smooth and bounded. Also, assume that $\Gamma = \partial\Omega \in C^1$, $v_0 \in H$, $f_0 \in L^2(0, \infty; L^2(\Omega))$ and $g_0 \in L^2(0, \infty; H^1(\Omega))$. Then the problem (1.1) has one global weak solution in time satisfying the the following energy inequality*

$$\begin{aligned}&\|v(t)\|^2 + 2\varepsilon \textstyle\int_0^t \|\nabla v(t)\|^2 dt \\ \le\ & \|v_0\|^2 + 2 \textstyle\int_0^t \{\int_\Gamma (k_0 v(t) + g_0(t)) \cdot v(t) d\Gamma + \int_\Omega f_0(t) \cdot v(t)) dx\} dt\end{aligned} \tag{2.3}$$

for any $0 \le t \le T$ and any given $T > 0$. Moreover, the global weak solution to the problem (1.1) is unique when $d = 2$.

Theorem 2. *Set $d = 2$ and let $\Omega = \mathcal{T} \times [-1, 1]$, where $\mathcal{T} = \frac{\mathcal{R}}{2\pi\mathcal{Z}}$ is a torus. Assume that $f_0 \in H^1(0, \infty; L^2(\Omega)) \bigcap L^2(0, \infty; H^1(\Omega))$ and $g_0 \in H^1(0, \infty; H^1(\Omega)) \bigcap L^2(0, \infty; H^2(\Omega))$. Also, assume that $v_0 \in V_1 \bigcap H^2(\Omega)$ satisfies the following zero order compatibility condition*

$$\varepsilon\frac{\partial v_0}{\partial n} - k_0 v_0 - g_0(0) = ((\varepsilon\frac{\partial v_0}{\partial n} - k_0 v_0 - g_0(0)) \cdot n)n, \quad x \in \partial\Omega = \mathcal{T} \times \{-1, 1\}, \tag{2.4}$$

where n is the unit outer normal vector of the boundary $\Gamma = \partial\Omega$. Then the problem (1.1) has one unique and global strong solution in time satisfying

$$\begin{aligned}&\partial_t v \in L^\infty(0, T; L^2(\Omega)) \bigcap L^2(0, T; V_1) \\ &v \in L^\infty(0, T; V_1) \bigcap L^2(0, T; H^2(\Omega)).\end{aligned} \tag{2.5}$$

Furthermore, if f_0, g_0, v_0 are smooth and satisfy suitable higher order compatibility conditions at the boundary, then the global-in-time weak solution to the problem (1.1) is smooth.

Theorem 3. *(**Viscosity vanishing limit** $\varepsilon \to 0$) Let $f_0(t), g_0^0(t), v_0^0$ be the given smooth functions and the assumptions in Theorem 1 hlod. Let $(v, p) = (v, p)(t)$ be one solution to the problem* (1.1) *given by Theorem 1, and let $(v^0, p^0) = (v^0, p^0)(t)$ defined on the interval $[0, T]$ be one smooth solution to the initial boundary value problem for 2D/3D incompressible Euler equations*

$$\begin{cases} v_t^0 + v^0 \cdot \nabla v^0 + \nabla p^0 = f_0, & x \in \Omega, 0 \le t \le T, \\ div\, v^0 = 0, & x \in \Omega, 0 \le t \le T, \\ p^0 = -\dfrac{1}{2}|v^0|^2 - g_0^0, & x \in \partial\Omega, 0 \le t \le T, \\ v(0, x) = v_0^0(x), & x \in \Omega. \end{cases} \tag{2.6}$$

Also, assume that $g_0(t) = -k_0 v^0(t) + g_0^0(t)n$ and $\|v_0 - v_0^0\| \to 0$ as $\varepsilon \to 0$. Then there exists one positive constant $K = K(T)$ such that, for any $k_0 \le -K$, it holds that $\|v(t) - v^0(t)\| \to 0$ for $0 \le t \le T$ when $\varepsilon \to 0$.

Remark 2. *If we replace the boundary condition* $(1.1)_3$ *by the boundary condition*

$$\varepsilon \frac{\partial v}{\partial n} - pn - \frac{1}{2}(v \cdot n)v = k_0 v + g_0, \quad x \in \partial\Omega, 0 \le t \le T,$$

then similar results to Theorem 1 and Theorem 2 hold. The similar results to Theorem 2 in the case of the general bounded smooth domain is also true, which will be discussed in another paper.

Remark 3. *If $f_0 = 0, \nabla g_0^0 = 0$, and the smooth function $v_0^0 \ne 0$ satisfies $div\, v_0^0 = 0, curl\, v_0^0 = 0$, $(v_0^0 \cdot n)|_\Gamma \ne 0$, and $(v_0^0 \times n)|_\Gamma \ne 0$, then $(v^0, p^0)(t, x) = (v_0^0, -\frac{1}{2}|v_0^0|^2 - g_0^0)(x)$ is one smooth solution to the system* (2.6)*. But, the well-posedness locally in time for the incompressible Euler system* (2.6) *with general initial data is open and will be remained to the future.*

2.2. Extension to incompressible Boussinesq equations

Definition 2. *(**The definition of the global weak solution to the problem** (1.2)) (v, p, ϱ) is called to be one global-in-time weak solution to the problem* (1.2)*, if for any $T > 0$, there exists the function pairs (v, p, ϱ), defined in the time interval $[0, T]$, satisfying $v \in L^\infty(0, T; L^2(\Omega)) \bigcap L^2(0, T; V_1)$ and $\varrho \in L^\infty(0, T; L^2(\Omega)) \bigcap L^2(0, T; H^1(\Omega))$ with $\partial_t v \in L^s(0, T; V_1')$ and $\partial_t \varrho \in L^s(0, T; (H^1(\Omega))')$ for some $s > 1$ satisfying*

$$\begin{aligned} &(\partial_t v(t), w)_\Omega + \varepsilon a_\Omega(v(t), w) + b_\Omega(v(t), v(t), w) - \frac{1}{2}\int_\Gamma |v(t)|^2 n \cdot w d\Gamma \\ &\quad = \int_\Gamma (k_0 v(t) + g_0(t)) \cdot w d\Gamma + (\varrho(t)e + f_0(t), w)_\Omega, w \in V_1, 0 \le t \le T, \\ &(\partial_t \varrho(t), \psi) + \kappa a_\Omega(\varrho(t), \psi) + b_\Omega(v(t), \varrho(t), \psi) - \frac{1}{2}\int_\Gamma (\varrho(t) v(t) \cdot n)\psi d\Gamma \\ &\quad = \int_\Gamma (k_1 \varrho(t) + g_1(t))\psi d\Gamma + (f_1(t), \psi)_\Omega, \psi \in H^1(\Omega), 0 \le t \le T, \\ &(v(0), \varrho(0)) = (v_0(x), \varrho_0(x)) \quad in\ V_1' \times (H^1(\Omega))'. \end{aligned} \tag{2.7}$$

Here $d\Gamma$ is the arc differential for $d = 2$ and the area differential for $d = 3$.

For the incompressible Boussinesq system, we state our results as follows.

Theorem 4. *Assume that the domain* $\Omega \subset \mathcal{R}^d, d = 2, 3$, *is bounded and smooth. Also, assume that the boundary* $\Gamma = \partial\Omega \in C^1$, $v_0 \in H$, $\varrho_0 \in L^2(\Omega)$, $f_0, f_1 \in L^2(0,\infty; L^2(\Omega))$ *and* $g_0, g_1 \in L^2(0,\infty; H^1(\Omega))$. *Then the problem* (1.2) *has one globally-in-time weak solution satisfying the following energy inequality*

$$\begin{aligned} &\|(\varrho(t), v(t))\|^2 + 2\varepsilon \int_0^t \|\nabla v(t)\|^2 dt + 2\kappa \int_0^t \|\nabla \varrho(t)\|^2 dt \\ \le\ & 2\int_0^t \{ \int_\Gamma (k_0 v(t) + g_0(t)) \cdot v(t) d\Gamma + \int_\Omega (\varrho(t) e + f_0(t)) \cdot v(t) dx \} dt \\ &+ 2\int_0^t \{ \int_\Gamma (k_1 \varrho(t) + g_1(t)) \varrho(t) d\Gamma + \int_\Omega f_1(t) \varrho(t)) dx \} dt, \quad 0 \le t \le T \end{aligned} \tag{2.8}$$

for any given positive T. *Moreover, the global-in-time weak solution to the problem* (1.2) *is unique when* $d = 2$.

Theorem 5. *Let* $\Omega = \mathcal{T} \times [-1, 1]$, *where* $\mathcal{T} = \frac{\mathcal{R}}{2\pi\mathcal{Z}}$ *is the torus. Assume that* $f_0, f_1 \in H^1(0,\infty; L^2(\Omega)) \bigcap L^2(0,\infty; H^1(\Omega))$ *and* $g_0, g_1 \in H^1(0,\infty; H^1(\Omega)) \bigcap L^2(0,\infty; H^2(\Omega))$. *Also, assume that* $v_0 \in V_1 \bigcap H^2(\Omega)$, $\varrho_0 \in H^2(\Omega)$ *satisfies the following zero order compatibility condition*

$$\begin{aligned} \varepsilon\tfrac{\partial v_0}{\partial n} - k_0 v_0 - g_0(0) = ((\varepsilon\tfrac{\partial v_0}{\partial n} - k_0 v_0 - g_0(0)) \cdot n) n, \quad x \in \partial\Omega = \mathcal{T} \times \{-1, 1\}, \\ \kappa\tfrac{\partial \varrho_0}{\partial n} - \tfrac{1}{2}\varrho_0 v_0 \cdot n = k_1 \varrho_0 + g_1(0), \quad x \in \partial\Omega = \mathcal{T} \times \{-1, 1\}, \end{aligned} \tag{2.9}$$

where n *is the unit outer normal vector of* $\Gamma = \partial\Omega$. *Then the problem* (1.2) *has one unique and globally-in-time strong solution satisfying*

$$\begin{aligned} &\partial_t v \in L^\infty(0, T; L^2(\Omega)) \bigcap L^2(0, T; V_1), \partial_t \varrho \in L^\infty(0, T; L^2(\Omega)) \bigcap L^2(0, T; H^1(\Omega)), \\ &v \in L^\infty(0, T; V_1) \bigcap L^2(0, T; H^2(\Omega)), \varrho \in L^\infty(0, T; H^1(\Omega)) \bigcap L^2(0, T; H^2(\Omega)). \end{aligned} \tag{2.10}$$

Furthermore, if the functions $f_0, f_1, g_0, g_1, \varrho_0, v_0$ *are smooth and satisfy suitable higher order compatibility conditions at the boundary, then the globally-in-time weak solution to the problem* (1.2) *is smooth.*

Remark 4. *We can not extend the present results in Theorems 3 and 4 to the case of* $\kappa = 0$ *and* $\varepsilon > 0$ *or the case of* $\kappa > 0$ *and* $\varepsilon = 0$ *because the corresponding initial boundary value problem for the inviscid or/and non-diffusive case for incompressible Boussinesq fluid in the bounded domain is different from* (1.2). *The discussion of these cases will be considered in the future.*

3. Proofs of our main Theorems

3.1. Proofs of Theorem 1 and Theorem 2

The proof of Theorem 1.

We prove Theorem 1 by constructing the approximating solution based on the Galerkin method and by establishing the a priori estimates. Let $T > 0$ be an arbitrarily given positive constant. By using

Lemma 3, there exists the orthogonal basis $\{e_l(x)\}_{l=1}^{\infty}$ of the space $V_1 \bigcap L^2(\Omega)$ with $\|e_l\| = 1$. For any given $l \geq 1$, we take the following l order approximating solution v^l of the solution v to the problem (1.1):

$$v^l(t) = v^l(t, x) = \sum_{j=1}^{m} g_j^l(t) e_j(x), l = 1, 2, \cdots, \tag{3.1}$$

where $g_j^l(t), j = 1, \cdots, l$, satisfies the system

$$\begin{aligned} &(\partial_t v^l(t), e_j)_\Omega + \varepsilon a_\Omega(v^l(t), e_j) + b_\Omega(v^l(t), v^l(t), e_j) - \frac{1}{2}\int_\Gamma |v^l(t)|^2 n \cdot e_j d\Gamma \\ &= \int_\Gamma (k_0 v^l(t) + g_0(t)) \cdot e_j d\Gamma + (f_0(t), e_j)_\Omega, j = 1, 2, \cdots, l, 0 \leq t \leq T, \\ &v^l(0) = v_0^l(x) = \sum_{k=1}^{l} v_0^k e_k(x), v_0(x) = \sum_{l=1}^{\infty} v_0^l e_l(x) \in V_1, \end{aligned} \tag{3.2}$$

which is one ordinary differential equations(ODE) for $g_j^l(t), j = 1, \cdots, l$. By standard existence and uniqueness theory for the ODE, there exist some $t^l > 0$ and one unique solution $\{g_j^l(t)\}_{j=1}^l$ of the system (3.2), defined in the interval $[0, t^l]$ and satisfying $g_j^l(t) \in H^1(0, t^l)$ and $\lim_{t \to t^l -} |g_j^l(t)| = \infty$ for some $j : 1 \leq j \leq l$. Now we want to prove $t^l = T$.

Step 1: The a priori $L^\infty(0, T; L^2(\Omega))$ estimate for $v^l(t, x)$.

Multiplying (3.2) by $g_j^l(t)$ and summing the resulting one from $j = 1$ to l, by using the fact that

$$b_\Omega(v^l, v^l, v^l) - \frac{1}{2}\int_\Gamma |v^l|^2 n \cdot v^l d\Gamma = 0$$

according to div $v^l = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|v^l(t)\|^2 + \varepsilon\|\nabla v^l(t)\|^2 = \int_\Gamma (k_0 v^l(t) + g_0) \cdot v^l(t) d\Gamma + (f_0(t), v^l(t))_\Omega, 0 \leq t \leq T. \tag{3.3}$$

With the help of the Young's inequality and the trace inequality (1.7), it follows from (3.3) that

$$\frac{1}{2}\frac{d}{dt}\|v^l(t)\|^2 + \frac{\varepsilon}{2}\|\nabla v^l(t)\|^2 \leq C\|v^l(t)\|^2 + \int_\Gamma |g_0(t)|^2 d\Gamma + \|f_0(t)\|^2, 0 \leq t \leq T. \tag{3.4}$$

By the Gronwall's inequality, we get from (3.4) that

$$\begin{aligned} &\|v^l(t)\|^2 + \varepsilon \int_0^t \|\nabla v^l(t)\|^2 dt \\ \leq\ & e^{Ct}(\|v_0\|^2 + \int_0^t \int_\Gamma |g_0(t)|^2 d\Gamma dt + \int_0^t \|f_0(t)\|^2 dt) \\ \leq\ & C(T) < \infty, \quad 0 \leq t \leq T, \end{aligned} \tag{3.5}$$

which implies that there exists a constant $C(T) > 0$ such that $|g_j^l(t)| \leq C(T) < \infty$ for any $l \geq 1$, $j = 1, 2, \cdots, l$, and $0 \leq t \leq t^l$, and, hence, $t^l = T$.

Step 2: The a priori $L^q(0,T;V_1')$ estimate for $\partial_t v^l(t,x)$.

For $j=1,2,\cdots,l$, noting the fact that $v^l|_\Gamma \neq 0$ and $e_j|_\Gamma \neq 0$, and by integration by parts, with the help of the Holder inequality and the trace inequalities (1.7), from (3.2) we get

$$\begin{aligned}
&|(\partial_t v^l(t), e_j)_\Omega| \\
= \quad & |-\varepsilon a_\Omega(v^l(t), e_j) + b_\Omega(v^l(t), e_j, v^l(t)) - \int_\Gamma (v^l(t)\cdot n)(v^l(t)\cdot e_j)d\Gamma \\
& -\frac{1}{2}\int_\Gamma |v^l(t)|^2 n\cdot e_j d\Gamma + \int_\Gamma (k_0 v^l(t) + g_0(t))\cdot e_j d\Gamma + (f_0(t), e_j)_\Omega| \\
\leq \quad & \varepsilon\|\nabla v^l(t)\|\|\nabla e_j\| + |b_\Omega(v^l(t), e^j, v^l(t))| \\
& + |-\int_\Gamma (v^l(t)\cdot n)(v^l(t)\cdot e_j)d\Gamma - \frac{1}{2}\int_\Gamma |v^l(t)|^2 n\cdot e_j d\Gamma| \\
& + C(\|v^l(t)\|_{H^1(\Omega)} + \|g_0(t)\|_{H^1(\Omega)})\|e_j\|_{H^1(\Omega)} + \|f_0(t)\|\|e_j\|.
\end{aligned} \tag{3.6}$$

To estimate the remaining integral terms in (3.6), we need the following basic inequality.

For any $w \in H^1(\Omega)$, one have

$$\|w\|_{L^4(\Omega)} \leq C\|w\|_{H^1(\Omega)}^{\frac{d}{4}}\|w\|^{1-\frac{d}{4}}, \quad d=2,3 \tag{3.7}$$

and

$$\|w|_\Gamma\|_{L^3(\Gamma)} \leq C\|w\|_{H^1(\Omega)}^{\frac{d+2}{6}}\|w\|^{1-\frac{d+2}{6}}, \quad d=2,3. \tag{3.8}$$

In fact, on one hand, by taking $q_1=4$ in (1.3) of Lemma 1 and using $\frac{1}{4}=\frac{1}{2}-\frac{s_1}{d}$, we get $H^{\frac{d}{4}}(\Omega)\hookrightarrow L^4(\Omega)$, which, together with the interpolation inequality ([36, 41]), yields

$$\|w\|_{L^4(\Omega)} \leq C\|w\|_{H^{\frac{d}{4}}(\Omega)} \leq C\|w\|_{H^1(\Omega)}^{\frac{d}{4}}\|w\|^{1-\frac{d}{4}}, \quad d=2,3,$$

which gives (3.7).

On the other hand, by taking $q_2=3$ in (1.4) of Lemma 1 and using $\frac{1}{3}=\frac{1}{2}-\frac{s_2}{d-1}$, we get $H^{\frac{d-1}{6}}(\Gamma)\hookrightarrow L^3(\Gamma)$, which, together with the trace inequality (1.5) and the interpolation inequality ([36, 41]), yields to

$$\|w|_\Gamma\|_{L^3(\Gamma)} \leq C\|w|_\Gamma\|_{H^{\frac{d-1}{6}}(\Gamma)} \leq C\|w\|_{H^{\frac{d-1}{6}+\frac{1}{2}}(\Omega)} \leq C\|w\|_{H^1(\Omega)}^{\frac{d+2}{6}}\|w\|^{1-\frac{d+2}{6}}, \quad d=2,3,$$

which gives (3.8).

By the Holder's inequality, using the fact that $\|n\|_{L^\infty} \leq C < \infty$ thanking to the smoothness of the domain, (3.7), (3.8), and the estimate (3.5), we get, for $j=1,2,\cdots,l$,

$$\begin{aligned}
& |b_\Omega(v^l(t), e^j, v^l(t))| + |-\int_\Gamma (v^l(t)\cdot n)(v^l(t)\cdot e_j)d\Gamma - \frac{1}{2}\int_\Gamma |v^l(t)|^2 n\cdot e_j d\Gamma| \\
\leq \quad & \|v^l(t)\|_{L^4(\Omega)}^2\|\nabla e_j\| + C\|v^l(t)|_\Gamma\|_{L^3(\Gamma)}^2\|e_j\|_{L^3(\Gamma)} \\
\leq \quad & C(T)(\|v^l(t)\|_{H^1(\Omega)}^{\frac{d}{2}} + \|v^l(t)\|_{H^1(\Omega)}^{\frac{d+2}{3}})\|e_j\|_{H^1(\Omega)}
\end{aligned} \tag{3.9}$$

Combining (3.6) and (3.9), using the Young's inequality and the fact that $1 \leq \frac{d}{2} \leq \frac{d+2}{3} < 2$ for $d = 2, 3$, we get, $0 \leq t \leq T$,

$$
\begin{aligned}
&|(\partial_t v^l(t), e_j)_\Omega| \\
\leq & C(T)(\|v^l(t)\|_{H^1(\Omega)} + \|v^l(t)\|_{H^1(\Omega)}^{\frac{d}{2}} + \|v^l(t)\|_{H^1(\Omega)}^{\frac{d+2}{3}} + \|g_0(t)\|_{H^1(\Omega)} + \|f_0(t)\|)\|e_j\|_{H^1(\Omega)} \\
\leq & C(T)(\|v^l(t)\|_{H^1(\Omega)}^{\frac{d+2}{3}} + \|g_0(t)\|_{H^1(\Omega)} + \|f_0(t)\| + C)\|e_j\|_{H^1(\Omega)},
\end{aligned}
$$

which gives

$$
\|\partial_t v^l(t)\|_{V_1'} \leq C(T)(\|v^l(t)\|_{H^1(\Omega)}^{\frac{d+2}{3}} + \|g_0(t)\|_{H^1(\Omega)} + \|f_0(t)\| + C), t \in [0, T], \tag{3.10}
$$

which implies that there exists a constant $C(T) > 0$ such that

$$
\|\partial_t v^l(t)\|_{L^q(0,T;V_1')} \leq C(T) < \infty, 1 \leq q = \frac{6}{d+2} \leq 2, \quad d = 2, 3. \tag{3.11}
$$

Combining the estimates (3.5) and (3.11), and using Lions-Aubin lemma, we have that there exists a subsequence, denoted still by $v^l(t)$, satisfying, when $l \to \infty$,

$$
\partial_t v^l(t) \rightharpoonup \partial_t v(t) \quad \text{weakly in } L^q(0, T; V_1'), \tag{3.12}
$$

$$
v^l(t) \rightharpoonup v(t) \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \tag{3.13}
$$

$$
v^l(t) \rightharpoonup v(t) \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \tag{3.14}
$$

$$
v^l(t) \to v(t) \quad \text{strongly in } L^2(0, T; H^{1-\delta}(\Omega)), \delta \in (0, \tfrac{1}{2}). \tag{3.15}
$$

Furthermore, according to the fact that the embedding $H^{1-\delta}(\Omega) \hookrightarrow L^2(\Gamma)$ is compact when $\delta < \frac{1}{2}$, we have

$$
v^l(t)|_\Gamma \to v(t)|_\Gamma \quad \text{strongly in } L^2(0, T; L^2(\Gamma)). \tag{3.16}
$$

Then, it follows from (3.15) and (3.16) that, for any $w \in V_1$, when $l \to \infty$,

$$
b_\Omega(v^l(t), v^l(t), w) \to b_\Omega(v(t), v(t), w) \quad \text{strongly in } L^1(0, T), \tag{3.17}
$$

and

$$
\frac{1}{2}\int_\Gamma |v^l(t)|^2 n \cdot w d\Gamma \to \frac{1}{2}\int_\Gamma |v(t)|^2 n \cdot w d\Gamma \quad \text{strongly in } L^1(0, T). \tag{3.18}
$$

Now, performing one limit as $l \to \infty$ in the equation (3.2) and using (3.12)–(3.18), we know that v satisfies the equation (2.1), and, then, integrating the equation (3.3) over $[0, T]$ with respect to the time t, setting $l \to \infty$ in the resulting one and using the convergence (3.15) and (3.16), we obtain the desired energy inequality (2.3) and the estimate

$$
\|v(t)\|^2 + \int_0^t \|\nabla v(t)\|^2 dt \leq C(T) < \infty, 0 \leq t \leq T. \tag{3.19}
$$

Step 3: Uniqueness of the solution when $d = 2$

Let v, v_* be any two solution to the problem (1.1). Then $\omega = v - v_*$ satisfies

$$\begin{aligned}
&(\partial_t \omega, w)_\Omega + \varepsilon a_\Omega(\omega, w) + b_\Omega(v, \omega, w) + b_\Omega(\omega, v, w) - b_\Omega(\omega, \omega, w) \\
&+ \frac{1}{2}\int_\Gamma |\omega|^2 n \cdot w d\Gamma - \int_\Gamma (\omega \cdot w)(n \cdot w) d\Gamma \\
=& \int_\Gamma k_0 \omega \cdot w d\Gamma, w \in V_1, 0 \leq t \leq T, \\
&\omega(0) = 0.
\end{aligned} \tag{3.20}$$

Taking $w = \omega(t)$ in the system (3.20), one have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 + \varepsilon \|\nabla \omega(t)\|^2 \\
=& - b_\Omega(\omega(t), v(t), \omega(t)) - \frac{1}{2}\int_\Gamma |\omega(t)|^2 n \cdot v(t) d\Gamma \\
&+ \int_\Gamma (\omega(t) \cdot v(t))(n \cdot \omega(t)) d\Gamma + k_0 \int_\Gamma |\omega(t)|^2 d\Gamma.
\end{aligned} \tag{3.21}$$

By the inequalities (3.7) and (3.8) with $d = 2$, and using (3.19), the Holder inequality, the Young inequality, and the trace inequality (1.7), we get

$$\begin{aligned}
&- b_\Omega(\omega(t), v(t), \omega(t)) \\
\leq& |\int_\Omega (\omega(t) \cdot \nabla) v(t) \cdot \omega(t) dx| \\
\leq& \|\nabla v(t)\| \|\omega(t)\|^2_{L^4(\Omega)} \\
\leq& C\|\nabla v(t)\| \|\omega(t)\|_{H^1(\Omega)} \|\omega(t)\| \\
\leq& \frac{\varepsilon}{8}\|\nabla \omega(t)\|^2 + C\|\omega(t)\|^2 + C\|\nabla v(t)\|^2 \|\omega(t)\|^2,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&- \frac{1}{2}\int_\Gamma |\omega(t)|^2 n \cdot v(t) d\Gamma + \int_\Gamma (\omega(t) \cdot v(t))(n \cdot \omega(t)) d\Gamma \\
\leq& C\int_\Gamma |\omega(t)|^2 |v(t)| d\Gamma \\
\leq& C\|v(t)|_\Gamma\|_{L^3(\Gamma)} \|\omega(t)|_\Gamma\|^2_{L^3(\Gamma)} \\
\leq& C\|v(t)\|^{\frac{2}{3}}_{H^1(\Omega)} \|v(t)\|^{\frac{1}{3}} \|\omega(t)\|^{\frac{4}{3}}_{H^1(\Omega)} \|\omega(t)\|^{\frac{2}{3}} \\
\leq& \frac{\varepsilon}{8}\|\nabla \omega(t)\|^2 + C\|\omega(t)\|^2 + C\|v(t)\|^2_{H^1(\Omega)} \|v(t)\| \|\omega(t)\|^2 \\
\leq& \frac{\varepsilon}{8}\|\nabla \omega(t)\|^2 + C(T)\|\omega(t)\|^2 + C(T)\|\nabla v(t)\|^2 \|\omega(t)\|^2
\end{aligned} \tag{3.23}$$

and

$$k_0 \int_\Gamma |\omega(t)|^2 d\Gamma \leq \tfrac{\varepsilon}{8}\|\nabla \omega(t)\|^2 + C\|\omega(t)\|^2. \tag{3.24}$$

Combining (3.21) with (3.22)–(3.24), we get

$$\frac{d}{dt}\|\omega(t)\|^2 + \varepsilon\|\nabla\omega(t)\|^2 \leq C\|\omega(t)\|^2 + C\|\nabla v(t)\|^2\|\omega(t)\|^2. \tag{3.25}$$

Using Gronwall's inequality to (3.25), $\omega(0) = 0$, and the inequality $\int_0^T \|\nabla v(t)\|^2 dt \leq C(T) < \infty$ which is given by (3.19), we obtain $\omega = 0$ for $0 \leq t \leq T$. This gives the uniqueness result on the global weak solution in the case of $d = 2$.

The proof of Theorem 1 is complete.

The proof of Theorem 2

Based on the proof of Theorem 1, so as to get Theorem 2, we need establish the more regularity estimates on the approximating solution $v^l(t)$ given by the system (3.1)-(3.2) when $d = 2$. To do this, we choose the functions $e_j(x)$, $j = 1, 2, \cdots$, to be the orthogonal basic function sequence of $V_1 \bigcap H^2(\Omega)$ and choose initial data $v_0^l(x) = v^l(0) = v^l(t = 0, x)$ satisfying

$$v_0^l \to v_0 \text{ strongly in } V_1 \bigcap H^2(\Omega)$$

and

$$\varepsilon\frac{\partial v_0^l}{\partial n} - k_0 v_0^l - g_0(0) = ((\varepsilon\frac{\partial v_0^l}{\partial n} - k_0 v_0^l - g_0(0)) \cdot n)n, \quad x \in \partial\Omega = \mathcal{T} \times \{-1, 1\}. \tag{3.26}$$

Step 1: The estimate for the initial data $\|\partial_t v^l(0)\|$ uniformly on l.

We obtain one estimate for $\|\partial_t v^l(0)\|$ uniformly on l by using the trace theorem given by Proposition 1. By (3.2), using integration by parts, (3.26), the Holder's inequality, and the trace property (1.10) in

Proposition 1 thanking to the fact that div $\partial_t v^l(0) = 0$ due to (3.1) and $e_j \in V_1$, we get

$$
\begin{aligned}
&\|\partial_t v^l(0)\|^2 \\
=&\varepsilon \int_\Omega \Delta v^l(0) \cdot \partial_t v^l(0) dx - \varepsilon \int_\Gamma \frac{\partial v^l(0)}{\partial n} \cdot \partial_t v^l(0) d\Gamma \\
&- b_\Omega(v^l(0), v^l(0), \partial_t v^l(0)) + \frac{1}{2} \int_\Gamma |v^l(0)|^2 n \cdot \partial_t v^l(0) d\Gamma \\
&+ \int_\Gamma (k_0 v^l(0) + g_0(0)) \cdot \partial_t v^l(0) d\Gamma + (f_0(0), \partial_t v^l(0))_\Omega \\
=&\varepsilon \int_\Omega \Delta v^l(0) \cdot \partial_t v^l(0) dx - b_\Omega(v^l(0), v^l(0), \partial_t v^l(0)) \\
&- \int_\Gamma ((\varepsilon \frac{\partial v^l(0)}{\partial n} - k_0 v^l(0) - g_0(0)) \cdot n)(n \cdot \partial_t v^l(0)) d\Gamma \\
&+ \frac{1}{2} \int_\Gamma |v^l(0)|^2 n \cdot \partial_t v^l(0) d\Gamma + (f_0(0), \partial_t v^l(0))_\Omega \\
\leq& \varepsilon \|\Delta v^l(0)\| \|\partial_t v^l(0)\| + \|v^l(0) \nabla v^l(0)\| \|\partial_t v^l(0)\| \\
&+ C \|(\frac{\partial v^l(0)}{\partial n}, v^l(0), g_0(0))|_\Gamma\|_{H^{\frac{1}{2}}(\Gamma)} \|(n \cdot \partial_t v^l(0))|_\Gamma\|_{H^{-\frac{1}{2}}(\Gamma)} \\
&+ C \||v^l(0)|_\Gamma|^2\|_{H^{\frac{1}{2}}(\Gamma)} \|(n \cdot \partial_t v^l(0))|_\Gamma\|_{H^{-\frac{1}{2}}(\Gamma)} + \|f_0(0)\| \|\partial_t v^l(0)\| \\
\leq& C \|\Delta v^l(0)\| \|\partial_t v^l(0)\| + C \|v^l(0)\|^2_{H^2(\Omega)} \|\partial_t v^l(0)\| \\
&+ C(\|\frac{\partial v^l(0)}{\partial n}\|_{H^1(\Omega)} + \|v^l(0)\|_{H^1(\Omega)} + \|g_0(0))\|_{H^1(\Omega)}) \|\partial_t v^l(0)\| \\
&+ C \|v^l(0)\|^2_{H^2(\Omega)} \|\partial_t v^l(0)\| + \|f_0(0)\| \|\partial_t v^l(0)\| \\
\leq& C(\|v_0\|_{H^2(\Omega)} + \|v_0\|^2_{H^2(\Omega)} + \|g_0(0)\|_{H^1(\Omega)} + \|f_0(0)\|) \|\partial_t v^l(0)\|,
\end{aligned}
\tag{3.27}
$$

which gives

$$
\|\partial_t v^l(t)|_{t=0}\| \leq C(\|v_0\|_{H^2(\Omega)} + \|v_0\|^2_{H^2(\Omega)} + \|g_0(0)\|_{H^1(\Omega)} + \|f_0(0)\|) \leq C < \infty, \tag{3.28}
$$

which means that, by using $(1.1)_1$ at $t = 0$,

$$
\|\partial_t v|_{t=0}\| = \|\varepsilon \Delta v_0 - v_0 \nabla v_0 - \nabla p(t = 0) + f_0(0)\| \leq C < \infty,
$$

where the pressure function $p(t = 0) = p_0 = p(0, x)$ is solved by

$$
\begin{cases}
-\Delta p_0 = \text{ div } (v_0 \cdot \nabla v_0 - f_0(0)), & x \in \Omega, \\
p_0 = \varepsilon \frac{\partial v_0}{\partial n} \cdot n - \frac{1}{2}|v_0|^2 - k_0 v_0 \cdot n - g_0(0) \cdot n, & x \in \Gamma = \partial\Omega.
\end{cases}
\tag{3.29}
$$

Thus, by comparing the condition $(3.29)_2$ and the boundary condition $(1.1)_3$, we know that the compatibility condition (2.4) is necessary.

Step 2: $L^\infty(0, T; L^2(\Omega))$ estimate for $\partial_t v^l(t)$ and $\nabla v^l(t)$.

Differentiating the system (3.2) with respect to t, multiplying the resulting one by $\partial_t g_j^l(t)$ and summing from $j = 1$ to l, by noticing that

$$b_\Omega(v^l, \partial_t v^l, \partial_t v^l) = \frac{1}{2}\int_\Gamma |\partial_t v^l|^2 n \cdot v^l(t) d\Gamma$$

due to div $v^l = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t v^l(t)\|^2 + \varepsilon\|\nabla\partial_t v^l(t)\|^2 \\
=& - b_\Omega(\partial_t v^l, v^l, \partial_t v^l) - \int_\Gamma [\frac{1}{2}|\partial_t v^l|^2 n \cdot v^l(t) - (v^l \cdot \partial_t v^l(t))(n \cdot \partial_t v^l(t))] d\Gamma \\
+& \int_\Gamma (k_0 \partial_t v^l(t) + \partial_t g_0(t)) \cdot \partial_t v^l(t) d\Gamma + (\partial_t f_0(t), \partial_t v^l(t))_\Omega, 0 \le t \le T.
\end{aligned} \tag{3.30}$$

Now we estimate each term in the right hand side of (3.30).

Using the inequality (3.7) with $d = 2$, with the help of the Holder's inequality and the Young's inequality, we get

$$\begin{aligned}
&|b_\Omega(\partial_t v^l, v^l, \partial_t v^l)| \\
\le& \|\nabla v^l(t)\| \|\partial_t v^l(t)\|^2_{L^4(\Omega)} \\
\le& C\|\nabla v^l(t)\| \|\partial_t v^l(t)\|_{H^1(\Omega)} \|\partial_t v^l(t)\| \\
\le& \frac{\varepsilon}{8}\|\nabla\partial_t v^l(t)\|^2 + C\|\partial_t v^l(t)\|^2 + C\|\nabla v^l(t)\|^2 \|\partial_t v^l(t)\|^2.
\end{aligned} \tag{3.31}$$

Using the fact that n is one constant unit vector, the inequality (3.8) with $d = 2$ and the estimate (3.5), with the help of the Holder's inequality and the Young's inequality, we get

$$\begin{aligned}
&\int_\Gamma [\frac{1}{2}|\partial_t v^l|^2 |n \cdot v^l(t)| + |(v^l \cdot \partial_t v^l(t))(n \cdot \partial_t v^l(t))|] d\Gamma \\
\le& C(\int_\Gamma |\partial_t v^l|^3 d\Gamma)^{\frac{2}{3}} (\int_\Gamma |v^l(t)|^3 d\Gamma)^{\frac{1}{3}} \\
\le& C\|\partial_t v^l(t)\|^{\frac{4}{3}}_{H^1(\Omega)} \|\partial_t v^l(t)\|^{\frac{2}{3}} \|v^l(t)\|^{\frac{2}{3}}_{H^1(\Omega)} \|v^l(t)\|^{\frac{1}{3}} \\
\le& \frac{\varepsilon}{8}\|\nabla\partial_t v^l(t)\|^2 + \frac{\varepsilon}{8}\|\partial_t v^l(t)\|^2 + C\|\partial_t v^l(t)\|^2 \|v^l(t)\|^2_{H^1(\Omega)} \|v^l(t)\| \\
\le& \frac{\varepsilon}{8}\|\nabla\partial_t v^l(t)\|^2 + C(T)\|\partial_t v^l(t)\|^2 + C(T)\|\nabla v^l(t)\|^2 \|\partial_t v^l(t)\|^2.
\end{aligned} \tag{3.32}$$

Also, by the trace inequality (1.7), with the help of the Holder's inequality and the Young's inequality, we get

$$\begin{aligned}
&\int_\Gamma (k_0 \partial_t v^l(t) + \partial_t g_0(t)) \cdot \partial_t v^l(t) d\Gamma + (\partial_t f_0(t), \partial_t v^l(t))_\Omega \\
\le& \frac{\varepsilon}{8}\|\nabla\partial_t v^l(t)\|^2 + C\|\partial_t v^l(t)\|^2 + C\|\partial_t g_0(t)\|^2_{H^1(\Omega)} + C\|\partial_t f_0(t)\|^2.
\end{aligned} \tag{3.33}$$

Combining (3.30) and (3.31)–(3.33), we get

$$\begin{aligned}&\frac{d}{dt}\|\partial_t v^l(t)\|^2+\varepsilon\|\nabla\partial_t v^l(t)\|^2\\ \leq& C(T)\|\partial_t v^l(t)\|^2+C(T)\|\nabla v^l(t)\|^2\|\partial_t v^l(t)\|^2\\ &+C\|\partial_t g_0(t)\|^2_{H^1(\Omega)}+C\|\partial_t f_0(t)\|^2, 0\leq t\leq T.\end{aligned}\tag{3.34}$$

Using the estimate (3.5), with the help of Gronwall's inequality, from (3.34), we get

$$\begin{aligned}&\|\partial_t v^l(t)\|^2\\ \leq& e^{C(T)\int_0^t(1+\|\nabla v^l(t)\|^2)dt}(\|\partial_t v^l(0)\|^2+C\int_0^t(\|\partial_t g_0(t)\|^2_{H^1(\Omega)}+\|\partial_t f_0(t)\|^2)dt)\\ \leq& C(T)<\infty, 0\leq t\leq T.\end{aligned}\tag{3.35}$$

Combining (3.34) and (3.35), we have

$$\begin{aligned}&\|\partial_t v^l(t)\|^2+\varepsilon\int_0^t\|\nabla\partial_t v^l(t)\|^2dt\\ \leq& C+C\int_0^t\|\partial_t g_0(t)\|^2_{H^1(\Omega)}dt+\int_0^t\|\partial_t f_0(t)\|^2dt\\ \leq& C(T)<\infty,\quad 0\leq t\leq T.\end{aligned}\tag{3.36}$$

Hence, by (3.4), using (3.5), (3.36), the assumptions on $f_0(t)$ and $g_0(t)$, and the Holder's inequality, we get

$$\|\nabla v^l(t)\|\leq C(T)<\infty,\quad 0\leq t\leq T.\tag{3.37}$$

Step 3: One $L^\infty(0,T;L^2(\Omega))$ estimate for $\partial_{x_1}v^l(t)$ and the boundary regularity estimate for $v^l(t)$ at $x_2=0$

In this case, we deal with the special domain $\Omega=\mathcal{T}\times[-1,1]$, where $\mathcal{T}=\frac{\mathcal{R}}{2\pi\mathcal{Z}}$. Thus, $\Gamma=\{x=(x_1,x_2)\in\mathcal{R}^2|x_1\in\mathcal{T}, x_2=-1, \text{ and } x_2=1\}$, $d\Gamma=ds=dx_1$ on $x\in\Gamma$, $n=\{0,1\}$ in the line $x_2=1$ and $n=\{0,-1\}$ in the line $x_2=-1$.

Denote $D_hv^l(t)=D_hv^l(t,x)=\frac{v^l(t,x_1+h,x_2)-v^l(t,x_1,x_2)}{h}$ and

$$D_{-h}D_hv^l(t,x)=\frac{v^l(t,x_1+h,x_2)-2v^l(t,x_1,x_2)+v^l(t,x_1-h,x_2)}{h^2}.$$

Using $(w,D_{-h}D_hw)=(D_hw,D_hw)$, $D_h(fg)=hD_hfD_hg+gD_hf+fD_hg$ and noting that $D_hn=0$ for one constant vector n, we get

$$\begin{aligned}&b_\Omega(v^l,v^l,D_{-h}D_hv^l)-\frac{1}{2}\int_\Gamma|v^l|^2n\cdot D_{-h}D_hv^md\Gamma\\ =&hb_\Omega(D_hv^l,D_hv^l,D_hv^l)+b_\Omega(D_hv^l,v^l,D_hv^l)+b_\Omega(v^l,D_hv^l,D_hv^l)\\ &-\frac{1}{2}\int_\Gamma(h|D_hv^l|^2+2v^l\cdot D_hv^l)n\cdot D_hv^ld\Gamma\\ =&b_\Omega(D_hv^l,v^l,D_hv^l)+\frac{1}{2}\int_\Gamma|D_hv^l|^2(n\cdot v^l)d\Gamma-\int_\Gamma(v^l\cdot D_hv^l)(n\cdot D_hv^l)d\Gamma.\end{aligned}\tag{3.38}$$

Taking $w = D_{-h}D_h v^l(t,x)$ in the equation (3.2) and using $(w, D_{-h}D_h w(t,x)) = (D_h w, D_h w)$ and (3.38), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|D_h v^l(t)\|^2 + \varepsilon\|\nabla D_h v^l(t)\|^2 \\
=& - b_\Omega(D_h v^l(t), v^l(t), D_h v^l(t)) - \frac{1}{2}\int_\Gamma |D_v u^l(t)|^2 (n\cdot v^l(t))d\Gamma \\
&+ \int_\Gamma (v^l(t)\cdot D_h v^l(t))(n\cdot D_h v^l(t))d\Gamma \\
&+ \int_\Gamma (k_0 D_h v^l(t) + D_h g_0(t))\cdot D_h v^l(t) d\Gamma + (D_h f_0(t), D_h v^l(t))_\Omega, 0\le t\le T, \\
&\|D_h v^l(0)\| \le \|\partial_{x_1} v_0(x)\| \le C < \infty, \text{ as } h\to 0.
\end{aligned} \tag{3.39}$$

Because each term in the right hand side of (3.39) is completely similar to that of (3.30), as one deal with each term in the right hand side of (3.30) in Step 2, by (3.39), with the help of (3.37) and the fact that $\|D_h v^l\| \le C\|\partial_{x_1} v^l\|$ for small h, we can get

$$\begin{aligned}
&\frac{d}{dt}\|D_h v^l(t)\|^2 + \varepsilon\|\nabla D_h v^l(t)\|^2 \\
\le& C(T)\|D_h v^l(t)\|^2 + C(T)\|\nabla v^l(t)\|^2\|D_h v^l(t)\|^2 \\
&+ C\|D_h g_0(t)\|^2_{H^1(\Omega)} + C\|D_h f_0(t)\|^2, 0\le t\le T.
\end{aligned} \tag{3.40}$$

By (3.40) and using the estimate (3.19), the fact that $\|D_h w\| \le C\|\partial_{x_1} w\|$, and the assumptions on $f_0(t)$ and $g_0(t)$, with the help of Gronwall's inequality, we have

$$\|D_h v^l(t)\|^2 + \varepsilon\int_0^t \|\nabla D_h v^l(t)\|^2 dt \le C(T) < \infty, \quad 0\le t\le T,$$

which gives

$$\|\partial_{x_1} v^l(t)\|^2 + \varepsilon\int_0^t \|\nabla \partial_{x_1} v^l(t)\|^2 dt \le C(T) < \infty, \quad t\in[0,T]. \tag{3.41}$$

This means that $\partial_{x_1} v^l(t,x) \in L^2(0,T;H^1(\Omega))$. Using the trace's theorem, we have $\partial_{x_1} v^l(t)|_\Gamma = \partial_{x_1} v^l(t, x_1, x_2 = \pm 1) \in L^2(0,T;H^{\frac{1}{2}}(\Gamma))$, i.e., $v^l(t)|_\Gamma \in L^2(0,T;H^{\frac{3}{2}}(\Gamma))$ and

$$\|v^l(t)|_\Gamma\|_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} \le C(T) < \infty. \tag{3.42}$$

Also, it is easy to verify that $v^l(t)$ satisfies the following incompressible NS system in the weak sense

$$\begin{cases}
-\varepsilon\Delta v^l + \nabla p^l = f_0 - \partial_t v^l - v^l\cdot\nabla v^l, x\in\Omega, t\in[0,T], \\
\text{div } v^l = 0, x\in\Omega, t\in[0,T], \\
v^l = v^l, x\in\Gamma, t\in[0,T].
\end{cases} \tag{3.43}$$

Applying Lemma 2 to (3.43), and using the Holder inequality and the Sobolev's embedding inequality $H^1(\Omega)\hookrightarrow L^4(\Omega)$, with the help of the Gagliardo-Nirenberg's inequality $\|Dv^l\|_{L^4(\Omega)} \le C_1\|v^l\|^{\frac{1}{4}}\|D^2 v^l\|^{\frac{3}{4}} +$

$C_2\|v^l\|$ for some constants $C_1 > 0$ and $C_2 > 0$ (see [41]), we have

$$\begin{aligned}
&\|v^l(t)\|_{H^2(\Omega)} + \|\nabla p^l(t)\|_{L^2(\Omega)} \\
\leq& C(\|(f_0 - \partial_t v^l - v^l \cdot \nabla v^l)\| + \|v^l(t)|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)}) \\
\leq& C(\|f_0(t)\| + \|\partial_t v^l(t)\| + \|v^l(t)\|_{L^4(\Omega)}\|\nabla v^l(t)\|_{L^4(\Omega)} + \|v^l(t)|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)}) \\
\leq& C(\|f_0(t)\| + \|\partial_t v^l(t)\| + \|v^l(t)\|_{H^1(\Omega)}\|v^l(t)\|^{\frac{1}{4}}\|v^l(t)\|^{\frac{3}{4}}_{H^2(\Omega)} + \|v^l(t)|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)}).
\end{aligned} \tag{3.44}$$

Using $f_0(t) \in L^2(0, T; L^2(\Omega))$ and the estimates (3.5), (3.36), (3.37), and (3.42), with the help of the Young's inequality, from (3.44) we obtain

$$\begin{aligned}
&\int_0^t \|v^l(t)\|^2_{H^2(\Omega)} dt \\
\leq& C\int_0^t (\|f(t)\|^2 + \|\partial_t v^l(t)\|^2 + \|v^l(t)\|^8_{H^1(\Omega)}\|v^l(t)\|^2)dt + C\int_0^t \|v^l(t)|_\Gamma\|^2_{H^{\frac{3}{2}}(\Gamma)} dt \\
\leq& C(T) < \infty, \quad 0 \leq t \leq T.
\end{aligned} \tag{3.45}$$

Now, performing the limit $l \to \infty$ in (3.36), (3.37), and (3.45), we obtain the estimate (2.5).

Similarly, if the initial data is smooth and satisfies the more high compatibility conditions, the more high regularity stated in Theorem 2 can also be established.

The proof of Theorem 2 is complete.

3.2. The proof of Theorem 3

We use so-called modified energy method. Firstly, it is easy to know that the strong solution to the system (2.6) satisfies

$$\begin{aligned}
&(\partial_t v^0(t), w)_\Omega + b_\Omega(v^0(t), v^0(t), w) - \frac{1}{2}\int_\Gamma |v^0(t)|^2 n \cdot w d\Gamma \\
&\quad = \int_\Gamma (g_0^0(t) n \cdot w d\Gamma + (f_0(t), w)_\Omega, w \in V_1, 0 \leq t \leq T, \quad v^0(0) = v_0^0(x).
\end{aligned} \tag{3.46}$$

Now take $w = v(t) - v^0(t) \in V_1$ in (3.46), we have

$$\begin{aligned}
&(\partial_t v^0(t), v(t) - v^0(t))_\Omega + b_\Omega(v^0(t), v^0(t), v(t) - v^0(t)) - \frac{1}{2}\int_\Gamma |v^0(t)|^2 n \cdot (v(t) - v^0(t)) d\Gamma \\
&\quad = \int_\Gamma (g_0^0(t) n \cdot (v(t) - v^0(t)) d\Gamma + (f_0(t), v(t) - v^0(t))_\Omega, 0 \leq t \leq T.
\end{aligned} \tag{3.47}$$

Then, combining (2.1) with $v = v(t) - v^0(t) \in V_1$ and (3.47), by the direct computation, and noting that $g_0(t) = -k_0 v^0(t) + g_0^0(t)n$ and the smoothness of $v^0(t)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |v(t) - v^0(t)|^2 dx \\
&= \int_\Omega (\partial_t v(t) - \partial_t v^0(t)) \cdot (v(t) - v^0(t)) dx \\
&= -\varepsilon \int_{\Omega_1} \nabla v(t) \cdot \nabla(v(t) - v^0(t)) dx - \int_{\Omega_1} (v(t)\nabla v(t) - v^0(t) \cdot \nabla v^0(t)) \cdot (v(t) - v^0(t)) dx \\
&\quad + \frac{1}{2}\int_{\Gamma_1} (|v(t)|^2 - |v^0(t)|^2) n \cdot (v(t) - v^0(t)) d\Gamma_1 \\
&\quad + \int_{\Gamma_1} [k_0(v(t) - v^0(t)) \cdot (v(t) - v^0(t)) d\Gamma_1 \\
&= -\varepsilon \int_\Omega |\nabla(v(t) - v^0(t))|^2 dx - \varepsilon \int_\Omega \nabla v^0(t) \cdot \nabla(v(t) - v^0(t)) dx \\
&\quad - \int_\Omega ((v(t) - v^0(t)) \cdot \nabla) v^0(t) \cdot (v(t) - v^0(t)) dx \\
&\quad + k_0 \int_\Gamma |v(t) - v^0(t)|^2 d\Gamma - \frac{1}{2}\int_\Gamma |v(t) - v^0(t)|^2 (v^0(t) \cdot n) d\Gamma \\
&\quad + \int_\Gamma ((v(t) - v^0(t)) \cdot v^0(t))((v(t) - v^0(t)) \cdot n) d\Gamma \\
&\leq -\varepsilon \int_\Omega |\nabla(v(t) - v^0(t))|^2 dx - \varepsilon \int_\Omega \nabla v^0(t) \cdot \nabla(v(t) - v^0(t)) dx \\
&\quad - \int_\Omega ((v(t) - v^0(t)) \cdot \nabla) v^0(t) \cdot (v(t) - v^0(t)) dx \\
&\quad + (k_0 + C(T)) \int_\Gamma |v(t) - v^0(t)|^2 d\Gamma
\end{aligned} \tag{3.48}$$

for some $C(T) > 0$. Thus, there exists a constant $K = K(T) > 0$ such that $k_0 + C(T) \leq 0$ for any $k_0 \leq -K$ and it holds, by using (3.48) and the Young's inequality, that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |v(t) - v^0(t)|^2 dx \\
&\leq -\frac{\varepsilon}{2}\int_\Omega |\nabla(v(t) - v^0(t))|^2 dx \\
&\quad + \|\nabla v^0(t,x)\|_{L^\infty([0,T]\times\Omega)} \int_\Omega |v(t) - v^0(t)|^2 dx \\
&\quad + \frac{\varepsilon}{2}\int_\Omega |\nabla v^0(t)|^2 dx
\end{aligned}$$

which yields to the desired convergence estimate

$$\int_\Omega |v(t) - v^0(t)|^2 dx \leq C(T)\int_\Omega |v_0 - v_0^0|^2 dx + C(T)\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

Here $C(T) > 0$ is one constant depending upon $v^0(t)$.

The proof of Theorem 3 is complete.

3.3. The proofs of Theorem 4 and Theorem 5

The proof of Theorem 4 are similar to that of Theorem 1 by constructing the following approximating solution (v^l, ϱ^l) of (v, ϱ) to the system (1.2):

$$v^l(t) = v^l(t, x) = \sum_{i=1}^{m} g_i^l(t)e_i(x), l = 1, 2, \cdots ,$$

$$\varrho^l(t) = \varrho^l(t, x) = \sum_{i=1}^{l} \varrho_i^l(t)\phi_i(x), l = 1, 2, \cdots ,$$

where $(g_i^l, \varrho_i^l)(t), i = 1, \cdots , l$, satisfies the ODEs

$$\begin{aligned}
&(\partial_t v^l(t), e_i)_\Omega + \varepsilon a_\Omega(v^l(t), e_i) + b_\Omega(v^l(t), v^l(t), e_i) - \frac{1}{2}\int_\Gamma |v^l(t)|^2 n \cdot e_i d\Gamma \\
&= \int_\Gamma (k_0 v^l(t) + g_0(t)) \cdot e_i d\Gamma + (\varrho^l(t)e + f_0(t), e_i)_\Omega, i = 1, 2, \cdots , l, 0 \le t \le T, \\
&(\partial_t \varrho^l(t), \phi_i)_\Omega + \kappa a_\Omega(\varrho^l(t), \phi_i) + b_\Omega(v^l(t), \varrho^l(t), \phi_i) - \frac{1}{2}\int_\Gamma \varrho^l(t)v^l(t) \cdot n\phi_i d\Gamma \\
&= \int_\Gamma (k_1\varrho^l(t) + g_1(t))\phi_i d\Gamma + (f_0(t), \phi_i)_\Omega, i = 1, 2, \cdots , l, 0 \le t \le T, \\
&v^l(0) = v_0^l(x) = \sum_{k=0}^{l} v_0^k e_k(x), v_0(x) = \sum_{k=0}^{\infty} v_0^k e_k(x) \in V_1, \\
&\varrho^l(0) = \varrho_0^l(x) = \sum_{k=0}^{l} \varrho_0^k \phi_k(x)), \varrho_0(x) = \sum_{k=0}^{\infty} \varrho_0^k \phi_k(x) \in H^1(\Omega),
\end{aligned}$$

and

$$-\kappa\frac{\partial\varrho^l(0)}{\partial n} + \frac{1}{2}\varrho^l(0)v^l(0) \cdot n + (k_1\varrho^l(0) + g_1(0)) = 0.$$

Here $\{e_i\}_{i=1}^\infty$ is chosen to be same as in the proof of Theorem 1 and $\{\phi_i\}_{i=1}^\infty$ is taken as an orthogonal basic function sequence of the Hilbert space $H^1(\Omega)$ with $\|\phi_i\| = 1$.

Also the proof of Theorem 5 are similar to that of Theorem 2 by using the following fact that

$$\begin{aligned}
\|\partial_t\varrho^l(0)\| =& -\kappa a_\Omega(\varrho^l(0), \partial_t\varrho^l(0)) - b_\Omega(v^l(0), \varrho^l(0), \partial_t\varrho^l(0)) \\
&+ \frac{1}{2}\int_\Gamma \varrho^l(0)v^l(0) \cdot n\partial_t\varrho^l(0)d\Gamma + \int_\Gamma (k_1\varrho^l(0) + g_1(0))\partial_t\varrho^l(0)d\Gamma \\
=&\kappa\int_\Omega \Delta\varrho^l(0)\partial_t\varrho^l(0)dx - b_\Omega(v^l(0), \varrho^l(0), \partial_t\varrho^l(0)) \\
&+ \int_\Gamma [-\kappa\frac{\partial\varrho^l(0)}{\partial n} + \frac{1}{2}\varrho^l(0)v^l(0) \cdot n + (k_1\varrho^l(0) + g_1(0))]\partial_t\varrho^l(0)d\Gamma \\
=&\kappa\int_\Omega \Delta\varrho^l(0)\partial_t\varrho^l(0)dx - b_\Omega(v^l(0), \varrho^l(0), \partial_t\varrho^l(0)) \\
\le& C(\|\varrho^l(0)\|_{H^2(\Omega)} + \|v^l(0)\|_{H^1(\Omega)}\|\varrho^l(0)\|_{H^2(\Omega)})\|\partial_t\varrho^l(0)\|
\end{aligned}$$

due to another compatibility condition for $\varrho_0(x)$ on $\partial\Omega$.

The details of the proofs of Theorem 4 and Theorem 5 are omitted.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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