Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 17(2): 550–581.
DOI: 10.3934/cam.2025022
Received: 13 November 2024
Revised: 08 May 2025
Accepted: 14 May 2025
Published: 21 May 2025

Research article

Blow-up for hyperbolized compressible Navier-Stokes equations

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Abstract: This work reports on initial value problems to hyperbolized compressible Navier-Stokes equations. Local existence and global existence for small data are addressed as well as singular limits for vanishing relaxation parameters. Then blow-up results for one- and multi-dimensional models are presented using ideas from T.C. Sideris from the 1980s.

Keywords: compressible Navier-Stokes equations; relaxation; hyperbolic systems; singular limits; blow-up
Mathematics Subject Classification: 35L60, 35B44

1. Introduction

The classical hyperbolic-parabolic system of the compressible Navier-Stokes (CNS) equations has been subject to a hyperbolization via a relaxation ansatz. The latter — known for classical heat conduction ever since Maxwell [1] in the 19th century or Cattaneo [2] in the 20th century introduced in order to avoid the phenomenon of infinite speed of propagation inherent in classical modeling of heat conduction — turns the system into a hyperbolic one. In view of the nonlinear character of CNS equations, the question of a possible blow-up of the solution is raised, since, roughly speaking, nonlinear hyperbolic systems tend to generate blow-ups in comparison to the corresponding parabolic ones. Here, *corresponding* means that the hyperbolized systems are characterized by a relaxation parameter $\tau > 0$, and formally turn into the original parabolic one for $\tau = 0$. See the linear example: the standard heat equation

$$\theta_t + \operatorname{div} q = 0, \qquad q + \nabla\theta = 0,$$

with temperature θ and heat flux q, leading to

$$\theta_t - \Delta\theta = 0,$$

turns with the relaxed/hyperbolized model

$$\theta_t + \operatorname{div} q = 0, \qquad \tau q_t + q + \nabla\theta = 0,$$

into

$$\tau\theta_{tt} + \theta_t - \Delta\theta = 0.$$

Naturally, the singular limit as $\tau \to 0$ is of interest, in particular for the nonlinear CNS equations to be discussed.

We will consider the following fully or partly hyperbolized models for compressible Navier-Stokes systems with heat conduction.

Model 1: ([3])

First, relaxing only in the heat conduction as above, we have in $\mathbb{R}^n \times [0, \infty)$ $(n = 1, 2, 3)$

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} S, \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \operatorname{div}(\rho u(e + \frac{1}{2}u^2) + up) + \operatorname{div} q = \operatorname{div}(uS), \end{cases} \tag{1.1}$$

where ρ, $u = (u_1, u_2, \cdots, u_n)$, p, S, e, and q represent fluid density, velocity, pressure, stress tensor, specific internal energy per unit mass, and heat flux, respectively. The equations $(1.1)_1$, $(1.1)_2$, and $(1.1)_3$ are the consequence of conservation of mass, momentum, and energy, respectively. To complete the system (1.1), we need to impose constitutive assumptions on p, S, e, and q. First, we assume the fluid to be a Newtonian fluid, that is,

$$S = \mu(\nabla u + \nabla u^T - \frac{2}{n}\operatorname{div} u\, I_n) + \lambda \operatorname{div} u\, I_n, \tag{1.2}$$

where μ and λ are positive constants. The heat flux q is assumed to satisfy

$$\tau\partial_t q + q + \kappa\nabla\theta = 0, \tag{1.3}$$

which represents Cattaneo's law (Maxwell's law, ...), and where θ denotes the absolute temperature. The pressure $p = p(\rho, \theta)$ and $e = e(\rho, \theta)$ satisfy

$$\rho^2 e_\rho(\rho, \theta) = p(\rho, \theta) - \theta p_\theta(\rho, \theta). \tag{1.4}$$

In particular, the case of a polytropic gas $p = R\rho\theta$, $e = c_v\theta$ is included here.

For the limit case $\tau = 0$, the system (1.1)–(1.3) is exactly the system of classical compressible Navier-Stokes equations, in which the relation between the heat flux and the temperature is governed by Fourier's law,

$$q = -\kappa\nabla\theta. \tag{1.5}$$

The well-posed theory has been widely studied for the system (1.1), (1.2) combined with Fourier's law (1.5), see [4–18]. In particular, the local existence and uniqueness of smooth solutions was established by Serrin [17] and Nash [16] for initial data far away from a vacuum. Later, Matsumura and Nishida [14] got global smooth solutions for small initial data without a vacuum. For large data, Xin [18], and Cho and Jin [4], showed that smooth solutions must blow up in finite time if the initial data has a vacuum state.

Although Fourier's law plays an important role in experimental and applied physics, it has the drawback of an inherent infinite propagation speed of signals. Cattaneo's (Maxwell's) law has been widely used in thermoelasticity which results in the second sound phenomenon, see [19–21] and the references cited therein. Note that it is not obvious that the results which hold for Fourier's law also hold for Cattaneo's law. Indeed, and for example, Fernández Sare and Racke [22] showed that, for certain Timoshenko-type thermoelastic system, Fourier's law preserves the property of exponential stability while Cattaneo's law destroys such a property.

Model 2: ([23])
Instead of relaxing in the heat equation as in Model 1, we take Fourier's law for the heat flux, but now Maxwell's relaxation for the stress tensor S, which replaces (1.2) by the differential equation

$$\tau\partial_t S + S = \mu(\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div}u\, I_n) + \lambda \mathrm{div}u\, I_n. \tag{1.6}$$

Here we will discuss a splitting of the tensor S, which was discussed by Yong [24] in the *isentropic* case leading to the following system with a revised Maxwell law, now for the *non-isentropic* case, that we are going to further investigate further.

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}(S_1) + \nabla S_2, \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \mathrm{div}(\rho u(e + \frac{1}{2}u^2) + up) - \kappa\triangle\theta = \mathrm{div}(u(S_1 + S_2 I_n)), \\ \tau_1\partial_t S_1 + S_1 = \mu(\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div}u I_n), \\ \tau_2\partial_t S_2 + S_2 = \lambda\mathrm{div}u, \end{cases} \tag{1.7}$$

where S_1 is an $n \times n$ square matrix, symmetric and traceless if it was initially, and S_2 is a scalar variable.

Pelton, Chakraborty, Malachosky, Guyot-Sionnest, and Sader [25] showed that such a "time lag", represented by $\tau_1, \tau_2 > 0$, cannot be neglected, even for simple fluids, in the experiments of high-frequency vibration of nano-scale mechanical devices immersed in water-glycerol mixtures. A similar revised Maxwell model was considered by Chakraborty and Sader [26] for a compressible viscoelastic fluid (isentropic case), where τ_1 counts for the shear relaxation time, and τ_2 counts for the compressional relaxation time. The importance of this model for describing high-frequency limits is underlined together with the presentation of numerical experiments. The authors conclude that it provides a general formalism to characterize the fluid-structure interaction of nanoscale mechanical devices vibrating in simple liquids.

Model 3: ([27])
Considering the two relaxations from Model 1 (resp., Model 2) in one space dimension simultaneously, and, additionally, reflecting Galilean invariance in the constitutive equations for these, we look at

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = S_x, \\ \rho e_t + \rho u e_x + p u_x + q_x = S u_x, \end{cases} \tag{1.8}$$

with

$$\tau_1(q_t + u q_x) + q + \kappa\theta_x = 0, \tag{1.9}$$

proposed by Christov and Jordan [28], and

$$\tau_2(S_t + uS_x) + S = \mu u_x. \tag{1.10}$$

Addtionally we specify the constitutive assumptions to

$$e = C_v\theta + \frac{\tau_1}{\kappa\theta\rho}q^2 + \frac{\tau_2}{2\mu\rho}S^2, \tag{1.11}$$

$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta}q^2 - \frac{\tau_2}{2\mu}S^2, \tag{1.12}$$

with positive constants C_v, R denoting the heat capacity at constant volume and the gas constant, respectively, such that they satisfy the thermodynamic equation (1.4). The dependence of the internal energy on q^2 is indicated by Coleman, Fabrizio, and Owen [29], where they rigorously prove that for heat equations with Cattaneo-type law, the formulation (1.20) is consistent with the second law of thermodynamics, see also [30–32].

Model 4: ([33])
For the results in dimensions $n = 2, 3$ having two relaxations, we consider the specialized model

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \rho\partial_t u + \rho u \cdot \nabla u + \nabla p = \mu \mathrm{div}(\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div} u I_n) + \nabla S_2, \\ \rho\partial_t e + \rho u \cdot \nabla e + p\mathrm{div} u + \mathrm{div} q = \mu(\nabla u + (\nabla u)^T - \frac{2}{n}\mathrm{div} u I_n) : \nabla u + S_2 \mathrm{div} u, \\ \tau_1(\partial_t q + u \cdot \nabla q) + q + \kappa\nabla\theta = 0, \\ \tau_3(\partial_t S_2 + u \cdot \nabla S_2) + S_2 = \lambda \mathrm{div} u, \end{cases} \tag{1.13}$$

where we have taken $\tau_2 = 0$ in (1.7). That is, we do not have a relaxation in S_1. This case seems to be mathematically not yet accessible, even locally. For the blow-up result in Section 4, we restrict the considerations to the limiting case $\mu = 0$. This restriction is not only motivated because it is mathematically accessible with respect to local existence and blow-up, but also with a physical background. In fact, there are recent studies determining the volume viscosity of a variety of gases which were found to be much larger (factor 10^4) than the corresponding shear viscosities, see [34]. For the blow-up result, we specify the constitutive equations,

$$e = C_v\theta + \frac{\tau_1}{\kappa\rho\theta}q^2 + \frac{\tau_3}{2\lambda\rho}S_2^2, \tag{1.14}$$

$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta}q^2 - \frac{\tau_3}{2\lambda}S_2^2. \tag{1.15}$$

Model 5: ([35])
For the second blow-up result in one space dimension, we modify Model 3, which was the basis for the first blow-up result, as follows.

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = S_x, \\ E_t + (uE + pu + q - Su)_x = 0, \end{cases} \tag{1.16}$$

where E represents the total energy,

$$\tau_1(\theta)(\rho q_t + \rho u \cdot q_x) + q + \kappa(\theta)\theta_x = 0, \tag{1.17}$$

and

$$\tau_2(\rho S_t + \rho u \cdot S_x) + S = \mu u_x. \tag{1.18}$$

The constitutive equation (1.18) was proposed by Freistühler [36, 37] for the isentropic case, see also Ruggeri [38] and Müller [39] for a similar model in the non-isentropic case.

Furthermore, we assume that the energy is given by

$$E = \frac{1}{2}\rho u^2 + \frac{\tau_2}{2\mu}\rho S^2 + \rho e(\theta, q), \tag{1.19}$$

and the specific internal energy e and the pressure p are given by

$$e(\theta) = C_v\theta + a(\theta)q^2, \qquad\qquad p(\rho, \theta) = R\rho\theta, \tag{1.20}$$

where

$$a(\theta) = \frac{Z(\theta)}{\theta} - \frac{1}{2}Z'(\theta) \quad \text{with} \quad Z(\theta) = \frac{\tau_1(\theta)}{\kappa(\theta)}.$$

The paper is organized as follows. In Section 2, we will recall results on the local well-posedness, small data global well-posedness, and the singular limit in finite time intervals. In Sections 3, 4, and 5, we will present blow-up results for one- and multi-dimensional models. Using ideas and techniques of Sideris from the 1980s [40, 41], the blow-up for some of the models above will be demonstrated by studying appropriate functionals that satisfy differential inequalities implying a blow-up of smooth solutions in finite time.

We introduce some notation. $W^{m,p} = W^{m,p}(\mathbb{R}^n)$, $0 \le m \le \infty$, $1 \le p \le \infty$, denotes the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}}$. H^m and L^p stand for $W^{m,2}(\mathbb{R}^n)$ and $W^{0,p}(\mathbb{R}^n)$, respectively.

2. Well-posedness and singular limit

Here, results on local well-posedness, on global well-posedness for small data, and on the singular limit $\tau \downarrow 0$ are presented for the Models 1–5. In addition to the governing differential equations, we always need initial conditions.

Starting with **Model 1** with differential equations (1.1)–(1.3), we have the initial conditions

$$(\rho(x, 0), u(x, 0), \theta(x, 0), q(x, 0)) = (\rho_0, u_0, \theta_0, q_0). \tag{2.1}$$

- Assumption A.1. The initial data satisfy

  $$\{(\rho_0, u_0, \theta_0, q_0)(x) : x \in \mathbb{R}^n\} \subset [\rho_*, \rho^*] \times [-C_1, C_1]^n \times [\theta_*, \theta^*] \times [-C_1, C_1]^n =: G_0,$$

  where $C_1 > 0$ as well as $0 < \rho_* < 1 < \rho^* < \infty$ and $0 < \theta_* < 1 < \theta^* < \infty$ are constants.

- Assumption A.2. Let $G := \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n$ denote the physical state space. For each given G_1 satisfying $G_0 \subset\subset G_1 \subset\subset G$, and for all $(\rho, u, \theta, q) \in G_1$, the pressure p and the internal energy e satisfy

$$p(\rho,\theta), p_\theta(\rho,\theta), p_\rho(\rho,\theta), e_\theta(\rho,\theta) > C(G_1) > 0, \tag{2.2}$$

where $C(G_1)$ is a positive constant depending on G_1.

Under these assumptions we have

Theorem 2.1 ([3]). *Let $n \geq 1$ and $s \geq s_0 + 1$, with $s_0 \geq [\frac{n}{2}] + 1$, be integers. Suppose that the assumptions A.1 and A.2 hold and that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0)$ are in H^s. Then, for each convex open subset G_1 satisfying $G_0 \subset\subset G_1 \subset\subset G$, there exists $T > 0$ such that the system* (1.1)*,* (1.3)*,* (2.1) *has a unique classical solution $(\rho^\tau, u^\tau, \theta^\tau, q^\tau)$ satisfying*

$$\begin{aligned}(\rho^\tau - 1, \theta^\tau - 1, q^\tau) &\in C([0,T], H^s) \cap C^1([0,T], H^{s-1}),\\ u^\tau &\in C([0,T], H^s) \cap C^1([0,T], H^{s-2})\end{aligned} \tag{2.3}$$

and

$$(\rho^\tau, u^\tau, \theta^\tau, q^\tau)(x,t) \in G_1, \quad \forall (x,t) \in \mathbb{R}^n \times [0,T].$$

We write the system (1.1) as a symmetric hyperbolic-parabolic system for for $\omega := (\rho, u, \theta, q)$,

$$A^0(\omega)\partial_t\omega + \Sigma_{j=1}^n A^j(\omega)\partial_{x_j}\omega - \Sigma_{j=1}^n B^{jk}(\omega)\partial^2_{x_jx_k}\omega + L(\omega)\omega = g(\omega, D_x\omega), \tag{2.4}$$

where

$$A^0(\omega) = \begin{pmatrix} \frac{c^2}{\rho} & 0 & 0 & 0 \\ 0 & \rho I_n & 0 & 0 \\ 0 & 0 & \frac{\rho e_\theta}{\theta} & 0 \\ 0 & 0 & 0 & \frac{\tau}{\kappa\theta} \end{pmatrix}, \ \Sigma_{j=1}^n A^j\xi_j = \begin{pmatrix} \frac{c^2}{\rho} u\cdot\xi & c^2\xi & 0 & 0 \\ c^2\xi^T & \rho(u\cdot\xi)I_n & p_\theta\xi^T & 0 \\ 0 & p_\theta\xi & \frac{\rho e_\theta}{\theta}u\cdot\xi & \frac{\xi}{\theta} \\ 0 & 0 & \frac{\xi^T}{\theta} & 0 \end{pmatrix},$$

$$\Sigma_{j=1}^n B^{jk}\xi_j\xi_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu I_n + (\mu+\mu')\xi^T\xi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ L(\omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa\theta} I_n \end{pmatrix},$$

$$g(\omega, D_x\omega) = \begin{pmatrix} 0 \\ 0 \\ \frac{\mu}{2\theta}|\nabla u + (\nabla u)^T|^2 + \frac{\mu'}{\theta}|\mathrm{div} u|^2 \\ 0 \end{pmatrix}, \ c^2 = p_\rho, \xi = (\xi_1, \cdots, \xi_n) \in \mathcal{S}^{n-1}.$$

The local existence theorem now follows from [42] (or see [43]). As the global existence result for small data, we have:

Theorem 2.2 ([3]). *Let $n \geq 2$ and $s \geq s_0 + 1$, with $s_0 \geq [\frac{n}{2}] + 1$, be integers. Suppose that $0 < \tau < \frac{2\kappa}{p_\theta(1,1)^2}$ and $(\rho_0 - 1, u_0, \theta_0 - 1, q_0) \in H^s$. Then there exists a positive constant δ such that if $\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0)\|_s \leq \delta$, then there exists a global unique solution $(\rho^\tau, u^\tau, \theta^\tau, q^\tau)$ of* (1.1)*,* (1.3)*,* (2.1) *satisfying*

$$\begin{aligned}(\rho^\tau - 1, \theta^\tau - 1, q^\tau) &\in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1}),\\ u^\tau &\in C([0,\infty), H^s) \cap C^1([0,\infty, H^{s-2}).\end{aligned} \tag{2.5}$$

For the proof, one linearizes the above system around the steady state $\bar{\omega} = (\bar{\rho}, \bar{u}, \bar{\theta}, \bar{q}) := (1, 0, 1, 0)$, and one has

$$A^0(\bar{\omega})\partial_t\omega + \Sigma A^j(\bar{\omega})\partial_{x_j}\omega - \Sigma B^{jk}(\bar{\omega})\partial^2_{x_jx_k}\omega + L(\bar{\omega})\omega = 0, \tag{2.6}$$

where

$$A^0(\bar{\omega}) = \begin{pmatrix} \bar{c}^2 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & \bar{e}_\theta & 0 \\ 0 & 0 & 0 & \frac{\tau}{\kappa} \end{pmatrix}, \ \Sigma A^j(\bar{\omega})\xi_j = \begin{pmatrix} 0 & \bar{c}^2\xi & 0 & 0 \\ \bar{c}^2\xi^T & 0 & \bar{p}_\theta\xi^T & 0 \\ 0 & \bar{p}_\theta\xi & 0 & \xi \\ 0 & 0 & \xi^T & 0 \end{pmatrix},$$

$$\Sigma B^{jk}(\bar{\omega})\xi_j\xi_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu I_n + (\mu+\mu')\xi^T\xi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ L(\bar{\omega}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa}I_n \end{pmatrix},$$

$$\bar{c} = c(1,1),\ \bar{p}_\theta = p_\theta(1,1),\ \bar{e}_\theta = e_\theta(1,1), \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathcal{S}^{n-1}.$$

We choose K^j such that

$$\Sigma K^j\xi_j = \alpha \begin{pmatrix} 0 & \bar{c}^2\xi & 0 & 0 \\ -\xi^T & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\kappa}{\tau}\xi \\ 0 & 0 & -\frac{\xi^T}{e_\theta} & 0 \end{pmatrix},$$

where $\alpha > 0$ will be chosen later. In order to satisfy the Kawashima conditions from [42], for proving the global existence for small data, one has to check that

$$\begin{aligned} &\frac{1}{2}\Sigma\left\{K^jA^k\xi_j\xi_k + (K^jA^k\xi_j\xi_k)^T\right\} + \Sigma B^{jk}\xi_j\xi_k + L \\ &= \begin{pmatrix} \alpha\bar{c}^4 & 0 & \frac{1}{2}\alpha\bar{p}_\theta\bar{c}^2 & 0 \\ 0 & \mu I_n + ((\mu+\mu') - \alpha\bar{c}^2)\xi^T\xi & 0 & -\alpha\frac{\bar{p}_\theta}{2\bar{e}_\theta}\xi^T\xi \\ \frac{1}{2}\alpha\bar{p}_\theta\bar{c}^2 & 0 & \alpha\frac{\kappa}{\tau} & 0 \\ 0 & -\alpha\frac{\bar{p}_\theta}{2\bar{e}_\theta}\xi^T\xi & 0 & \frac{1}{\kappa}I_n - \alpha\frac{1}{\bar{e}_\theta}\xi^T\xi \end{pmatrix} \end{aligned}$$

is a positive definite matrix for any $\xi \in \mathcal{S}^{n-1}$, which holds true for sufficiently small α.

To show the convergence of the relaxed system ($\tau > 0$) to the classical CNS equations ($\tau = 0$), we assume the natural compatibility condition $q_0 = -\kappa\nabla\theta_0$. Let G_1 be given satisfying $G_0 \subset\subset G_1 \subset\subset G$. Define $T_\tau = \sup\{T > 0; (\rho^\tau - 1, v^\tau, \theta^\tau - 1, q^\tau) \in C([0,T], H^s), \quad (\rho^\tau, v^\tau, \theta^\tau, q^\tau)(x,t) \in G_1, \quad \forall (x,t) \in \mathbb{R}^n \times [0,T]\}$.

Theorem 2.3 ([3]). *Let (ρ, u, θ) be a smooth solution to the classical compressible Navier-Stokes equations with $(\rho(x,0), u(x,0), \theta(x,0)) = (\rho_0, u_0, \theta_0)$ satisfying*

$$\rho \in C([0,T_*], H^{s+3}) \cap C^1([0,T_*], H^{s+2}),\ (u,\theta) \in C([0,T_*], H^{s+3}) \cap C^1([0,T_*], H^{s+1})$$

with $T_* > 0$ finite. Then there are positive constants τ_0 and C such that for $\tau \le \tau_0$,

$$\|(\rho^\tau, u^\tau, \theta^\tau)(t,\cdot) - (\rho, u, \theta)(t,\cdot)\|_s \le C\tau \tag{2.7}$$

and

$$\|(q^\tau + \kappa\nabla\theta)(t,\cdot)\|_s \leq C\tau^{\frac{1}{2}} \tag{2.8}$$

for $t \in [0, \min\{T_*, T_\tau\})$, *where* C *does not depend on* τ*. In particular,* T_τ *is independent of* τ*.*

For the proof, we introduce the variable $q := -\kappa\nabla\theta$ and define

$$\rho^d := \frac{\rho^\tau - \rho}{\tau}, u^d = \frac{u^\tau - u}{\tau}, \theta^d = \frac{\theta^\tau - \theta}{\tau}, q^d = \frac{q^\tau - q}{\tau}. \tag{2.9}$$

Lengthy energy estimates give, for small τ and for $t < \min\{T_*, T_\tau\}$,

$$\|(\rho^d, u^d, \theta^d)(t,\cdot)\|_s \leq C, \quad \|\sqrt{\tau}q^d(t,\cdot)\|_s \leq C, \tag{2.10}$$

where $C > 0$ denotes constants not depending on τ or t.

Looking at **Model 2** with differential equations (1.7), we have the initial conditions

$$(\rho(x,0), u(x,0), \theta(x,0), S_1(x,0), S_2(x,0)) = (\rho_0, u_0, \theta_0, S_{10}, S_{20})). \tag{2.11}$$

Assumptions analogous to Assumptions A.1 and A.2 in (2.2) are assumed to hold. Then we have the following local existence theorem.

Theorem 2.4 ([23])**.** *Let* $s \geq s_0 + 1$ *with* $s_0 \geq [\frac{n}{2}] + 1$ *be integers. Suppose that the initial data* $(\rho_0 - 1, u_0, \theta_0 - 1, S_{10}, S_{20})$ *are in* H^s*. Then, for each convex open subset* G_1 *satisfying* $G_0 \subset\subset G_1 \subset\subset G$*, there exists* $T_{ex} > 0$ *such that the system* (1.7)*,* (2.11) *has a unique classical solution* $(\rho, u, \theta, S_1, S_2)$ *satisfying*

$$\begin{cases} (\rho - 1, u, S_1, S_2) \in C([0, T_{ex}], H^s) \cap C^1([0, T_{ex}], H^{s-1}), \\ \theta - 1 \in C([0, T_{ex}], H^s) \cap C^1([0, T_{ex}], H^{s-2}) \end{cases} \tag{2.12}$$

and

$$(\rho, u, \theta, S_1, S_2)(x,t) \in G_1, \quad \forall (x,t) \in \mathbb{R}^n \times [0, T_{ex}].$$

For the proof, a similar strategy as in the proof of Theorem 2.1 in Model 1 is applicable, i.e., transforming, after linearizing around a constant state, the system to a symmetric hyperbolic-parabolic one. In the two-dimensional case $n = 2$, one can easily check that the system can be written in a symmetric form immediately, while in the 3-d case one needs further transformations to get a system in a symmetric form, see [23].

Using the explicit symmetrizer, one can check Kawashima's conditions, yielding the following global existence theorem for small data.

Theorem 2.5 ([23])**.** *Let* $s \geq s_0 + 1$ *with* $s_0 \geq [\frac{n}{2}] + 1$ *be integers. Suppose that the initial data satisfy* $(\rho_0 - 1, u_0, \theta_0 - 1, S_{10}, S_{20}) \in H^s$*. Then there exists a positive constant* δ *such that if* $\|(\rho_0 - 1, u_0, \theta_0 - 1, S_{10}, S_{20})\|_s \leq \delta$*, there exists a global unique solution* $(\rho, u, \theta, S_1, S_2)$ *to the system* (1.7)*,* (2.11) *satisfying*

$$\begin{cases} (\rho - 1, u, S_1, S_2) \in C([0, \infty), H^s) \cap C^1([0, \infty), H^{s-1}), \\ (\theta - 1) \in C([0, \infty), H^s) \cap C^1([0, \infty, H^{s-2}). \end{cases} \tag{2.13}$$

We remark that Kawashima's results also imply decay properties of the solutions, that is,

$$\|(\rho-1,u,\theta-1,S_1,S_2)\|_{s-(s_0+1)} \to 0, \qquad \text{as} \quad t\to\infty.$$

Moreover, for $n=3$, if we further assume $s \geq s_0+2$ and $\|(\rho-1,u,\theta-1,S_1,S_2)\|_{L^p} \leq \delta$ where $p\in[1,\frac{3}{2}]$, then the solutions have the following decay:

$$\|(\rho-1,u,\theta-1,S_1,S_2)\|_{s-1} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}\|(\rho_0-1,u_0,\theta_0-1,S_{10},S_{20})\|_{s-1,p},$$

where the constant C is neither depending on t nor on the data.

The compatibility of the revised Maxwell law with the Newtonian law in terms of the limit $\tau_1 = \tau_2 =: \tau \downarrow 0$ is described in the next theorem, where the following natural compatibility conditions on the initial data are assumed:

$$S_{10} = \mu(\nabla u_0 + (\nabla u_0)^T - \frac{2}{n}\mathrm{div} u_0\, I_n),\ S_{20} = \lambda \mathrm{div} u_0. \tag{2.14}$$

Denote by $(\rho^\tau,u^\tau,\theta^\tau,S_1^\tau,S_2^\tau)$ the solutions given by Theorem 2.4 with G_1 satisfying $G_0 \subset\subset G_1 \subset\subset G$. Denoting

$$T_\tau = \sup\{T>0, (\rho^\tau-1,u^\tau,\theta^\tau-1,S_1^\tau,S_2^\tau)\in C([0,T],H^s), (\rho^\tau,u^\tau,\theta^\tau,S_1^\tau,S_2^\tau)\in G_1\},$$

we have:

Theorem 2.6 ([23]). *Let (ρ,u,θ) be a smooth solution to the classical compressible Navier-Stokes equations with $(\rho(x,0),u(x,0),\theta(x,0)) = (\rho_0,u_0,\theta_0)$ satisfying*

$$\rho\in C([0,T_*],H^{s+3})\cap C^1([0,T_*],H^{s+2}),\ (u,\theta)\in C([0,T_*],H^{s+3})\cap C^1([0,T_*],H^{s+1})$$

with $T_*>0$ (finite). Then there are positive constants τ_0 and C such that for $\tau\leq\tau_0$,

$$\|(\rho^\tau,u^\tau,\theta^\tau)(t,\cdot)-(\rho,u,\theta)(t,\cdot)\|_s \leq C\tau \tag{2.15}$$

and

$$\|S_1^\tau(t,\cdot)-\mu\left(\nabla u+(\nabla u)^T-\frac{2}{n}\mathrm{div} u I_n\right)(t,\cdot)\|_s + \|S_2^\tau(t,\cdot)-\lambda\mathrm{div} u(t,\cdot)\|_s \leq C\tau^{\frac{1}{2}} \tag{2.16}$$

for $t\in[0,\min\{T_*,T_\tau\})$, where C does not depend on τ.

Introducing the variables $S_1^0 := \mu(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div} u I_n), S_2^0 := \lambda\,\mathrm{div} u$ and defining

$$\rho^d := \frac{\rho^\tau-\rho}{\tau}, u^d := \frac{u^\tau-u}{\tau}, \theta^d := \frac{\theta^\tau-\theta}{\tau}, S_1^d := \frac{S_1^\tau-S_1^0}{\tau}, S_2^d := \frac{S_2^\tau-S_2^0}{\tau}, \tag{2.17}$$

the aim is to show that, for small τ and for $t<\min\{T_*,T_\tau\}$,

$$\|(\rho^d,u^d,\theta^d)(t,\cdot)\|_s \leq C, \quad \|\sqrt{\tau}(S_1^d,S_2^d)(t,\cdot)\|_s \leq C, \tag{2.18}$$

where $C>0$ denotes constants not depending on τ or t. This is achieved using the energy method combined with sophisticated estimates of the nonlinear terms.

For **Model 3** in one space dimension with differential equations (1.8)–(1.10), where we consider two relaxations with nonlinearities in the relaxed equations reflecting the Galilean invariance, we have the initial conditions

$$(\rho(x,0), u(x,0), \theta(x,0), q(x,0), S(x,0)) = (\rho_0, u_0, \theta_0, q_0, S_0). \tag{2.19}$$

We recall that the internal energy e and the pressure p are assumed to have the form (1.11) (resp., (1.12)) and satisfy the thermodynamic equation $\rho^2 e_\rho = p - \theta p_\theta$. Using this we may rewrite the differential equations as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_\rho \rho_x + p_\theta \theta_x + p_q q_x + (p_S - 1) S_x = 0, \\ \rho e_\theta \theta_t + (\rho u e_\theta - \frac{2q}{\theta})\theta_x + \theta p_\theta u_x + q_x = \frac{2q^2}{\kappa\theta} + \frac{S^2}{\mu}, \\ \tau_1(q_t + u q_x) + q + \kappa\theta_x = 0, \\ \tau_2(S_t + u S_x) + S = \mu u_x. \end{cases} \tag{2.20}$$

The system (2.20) is not symmetric. But one can show that there exists a δ such that if $|(\rho-1, \theta-1, q, S)| < \delta$, then the system (2.20) is strictly hyperbolic, since for the first-order system for $V := (\rho, u, \theta, q, S)'$, given by

$$V_t + A(V)\partial_x V + B(V)V = F(V), \tag{2.21}$$

where

$$A(V) = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ \frac{p_\rho}{\rho} & u & \frac{p_\theta}{\rho} & \frac{p_q}{\rho} & \frac{p_S - 1}{\rho} \\ 0 & \frac{\theta p_\theta}{\rho e_\theta} & u - \frac{2q}{\rho\theta e_\theta} & 1 & 0 \\ 0 & 0 & \frac{\kappa}{\tau_1} & u & 0 \\ 0 & -\frac{\mu}{\tau_2} & 0 & 0 & u \end{pmatrix}, B(V) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_1} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\tau_2} \end{pmatrix} \tag{2.22}$$

and $F(V) := \left(0, 0, \frac{2q^2}{\kappa\theta} + \frac{S^2}{\mu}, 0, 0\right)'$, the eigenvalues of the matrix $A(V)$ are then real and distinct. The following local existence theorem then follows, see [47], and it also implies that (2.20) is symmetrizable.

Theorem 2.7 ([27]). *Let $s \geq 2$. Then there is $\delta > 0$ such that for $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0) \in W^{s,2}(\mathbb{R})$ with $\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)\|_{s,2} < \delta$, there exists a unique local solution (ρ, u, θ, q, S) to the system* (1.8)–(1.10), (2.19) *in some time interval* $[0, T]$ *with*

$$(\rho - 1, u, \theta - 1, q, S) \in C^0([0,T], H^s(\mathbb{R})) \cap C^1([0,T], H^{s-1}(\mathbb{R})). \tag{2.23}$$

The global well-posedness for small data is given by:

Theorem 2.8 ([27]). *There exists $\varepsilon > 0$ such that if*

$$\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0\|_{H^2}^2 < \varepsilon^2, \tag{2.24}$$

there exists a global solution $(\rho, u, \theta, q, S)(x, t) \in C^1([0, +\infty) \times \mathbb{R})$ *to the system* (1.8)–(1.10)*,* (2.19) *satisfying*

$$\frac{3}{4} \leq \sup_{x,t}(\rho(x,t), \theta(x,t)) \leq \frac{5}{4}$$

and

$$\sup_{t\in[0,\infty)} \|(\rho - 1, u, \theta - 1, q, S)\|_{H^2}^2 \leq C\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0\|_{H^2}^2 \leq C\varepsilon^2, \tag{2.25}$$

where C is a constant which is independent of ε*. Moreover, the solution converges uniformly in* $x \in \mathbb{R}$ *to the constant state* $(1, 0, 1, 0, 0)$ *as* $t \to \infty$*. Namely,*

$$\|(\rho - 1, u, \theta - 1, q, S)\|_{L^\infty} + \|(\rho_x, u_x, \theta_x, q_x, S_x)\|_{L^2} \to 0 \qquad \text{as} \quad t \to \infty.$$

For the proof, a series of a priori estimates for the local solution is derived, using the energy functional

$$\begin{aligned} E(t) := \sup_{0\leq s\leq t} \|(\rho - 1, u, \theta - 1, q, S)(s,\cdot)\|_{H^2}^2 + \sup_{0\leq s\leq t} \|(\rho_t, u_t, \theta_t, q_t, S_t)\|_{H^1}^2 \\ + \int_0^t \|(\rho_x, \rho_t, u_x, u_t, \theta_x, \theta_t, q_x, q_t, q, S_x, S_t, S)(s,\cdot)\|_{H^1}^2 \mathrm{d}s \end{aligned} \tag{2.26}$$

and the equality

$$\begin{aligned} &\left[c_v\rho(\theta - \ln\theta - 1) + R(\rho\ln\rho - \rho + 1) + (1 - \frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2 + \frac{1}{2}\rho u^2 + \frac{\tau_2}{2\mu}S^2\right]_t \\ &+ [\rho u c_v(\theta - \ln\theta - 1) + u(1 - \frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2 + \frac{\tau_2}{2\mu}uS^2 + R\rho u\ln\rho - R\rho u \\ &- \frac{q}{\theta} + \frac{1}{2}\rho u^3 + pu + q - Su]_x + \frac{q^2}{\kappa\theta^2} + \frac{S^2}{\theta\mu} = 0, \end{aligned} \tag{2.27}$$

finally allowing to continue a local solution.

For a description of the singular limit, we assume $\tau_1 = \tau_2 =: \tau$ and the compatibility condition

$$S_0 = \mu(u_0)_x, \; q_0 = -\kappa(\theta_0)_x.$$

Let $(\rho^\tau, u^\tau, \theta^\tau, q^\tau, S^\tau)$ be solutions given by Theorem 2.7. Define

$$\begin{aligned} T_\tau \;=\; &\sup\{T > 0; (\rho^\tau - 1, u^\tau, \theta^\tau - 1, q^\tau, S^\tau) \in C([0,T], H^2), \rho^\tau > 0, \theta^\tau > 0, \\ &\forall (x,t) \in \mathbb{R}^n \times [0,T]\}. \end{aligned}$$

Theorem 2.9 ([27])**.** *Let* (ρ, u, θ) *be the smooth solution to the classical compressible Navier-Stokes equations with* $(\rho(x,0), u(x,0), \theta(x,0)) = (\rho_0, u_0, \theta_0)$ *satisfying*

$$\inf_{(x,t)\in\mathbb{R}\times[0,T_*]}(\rho(x,t), \theta(x,t)) > 0$$

and

$$(\rho - 1) \in C([0,T_*], H^5) \cap C^1([0,T_*], H^4),$$

$$(u, \theta - 1) \in C([0, T_*], H^5) \cap C^1([0, T_*], H^3),$$

with finite $T_* > 0$. Then, there exist constants τ_0 and C such that for $\tau \le \tau_0$,

$$\|(\rho^\tau, u^\tau, \theta^\tau)(t, \cdot) - (\rho, u, \theta)(t, \cdot)\|_{H^2} \le C\tau, \tag{2.28}$$

and

$$\|(q^\tau + \kappa\theta_x, S^\tau - \mu u_x)\|_{H^2} \le C\tau^{\frac{1}{2}}, \tag{2.29}$$

for all $t \in (0, min(T_*, T_\tau))$, and the constant C is independent of τ.

The proof again looks at the differential equations for the differences $\rho^d = \frac{\rho^\tau - \rho}{\tau}$, $u^d = \frac{u^\tau - u}{\tau}$, $\theta^d = \frac{\theta^\tau - \theta}{\tau}$, $q^d = \frac{q^\tau - q}{\tau}$, $S^d = \frac{S^\tau - S}{\tau}$, where $q = -\kappa\theta_x$ and $S = \mu u_x$. For small τ and $t < \min\{T_*, T_\tau\}$, one proves

$$\|(\rho^d, u^d, \theta^d)(t, \cdot)\|_{H^2} \le C, \quad \|\sqrt{\tau}(q^d, S^d)(t, \cdot)\|_{H^2} \le C, \tag{2.30}$$

with $C > 0$ not depending on τ. Here, on a technical level, the H^5-regularity is needed.

For **Model 4** in higher dimensions with differential equations (1.13) and initial conditions

$$(\rho(x, 0), u(x, 0), \theta(x, 0), q(x, 0), S_2(x, 0)) = (\rho_0, u_0, \theta_0, q_0, S_{20}), \tag{2.31}$$

we distinguish the cases $\mu > 0$ and $\mu = 0$.

For $\mu > 0$, the local existence theorem reads:

Theorem 2.10 ([33]). *Suppose that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20}) \in H^3$. Then there exists $T = T(\|(\rho_0, \ldots, S_{20})\|_{H^3}) > 0$, such that the system* (1.13)*,* (2.31) *has an unique classical solution $(\rho, u, \theta, q, S_2)$ satisfying*

$$\begin{aligned}(\rho - 1, \theta - 1, q, S_2) &\in C([0, T], H^3) \cap C^1([0, T], H^2)\\ u &\in C([0, T], H^3) \cap C^1([0, T], H^1).\end{aligned}$$

The proof rewrites the system as a symmetric hyperbolic-parabolic one and uses the results of Kawashima, see [42] or [44].

For the global existence for small data let

$$\begin{aligned}En(t) := \sup_{0 \le \tau \le t} &\|(\rho - 1, u, \theta - 1, q, S_2)(\tau, \cdot)\|_{H^3}^2 +\\ &\int_0^t \left(\|(\nabla\rho, \nabla\theta)\|_{H^2}^2 + \|(q, S_2)\|_{H^3}^2 + \|\nabla u\|_{H^3}^2\right) \mathrm{d}t.\end{aligned} \tag{2.32}$$

Then we have:

Theorem 2.11 ([33]). *Let $\tau_1 > 0, \tau_2 = 0, \tau_3 > 0$, and $\mu > 0$. Suppose for the initial data*

$$(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20}) \in H^3.$$

Then, there exists a small constant $\delta > 0$ such that if $En(0) < \delta$, then the system (1.13)*,* (2.31) *has a unique solution $(\rho, u, \theta, q, S_2)$ globally in time such that $(\rho - 1, u, \theta - 1, q, S_2) \in C(0, +\infty; H^3)$, $(\nabla\rho, \nabla\theta) \in L^2(0, +\infty; H^2)$, $\nabla u \in L^2(0, +\infty; H^3)$, $(q, S_2) \in L^2(0, +\infty; H^3)$.*

For any $t > 0$, we have

$$\|(\rho-1,u,\theta-1,q,S_2)\|_{H^3}^2+\int_0^t\Big(\|(\nabla\rho,\nabla\theta)\|_{H^2}^2+\|\nabla u\|_{H^3}^2+\|(q,S_2)\|_{H^3}^2\Big)\,\mathrm{d}t \le CEn(0), \tag{2.33}$$

where C is a constant being independent of t and of the initial data. Moreover, the solution decays in the sense

$$\|\nabla(\rho,u,\theta,q,S_2)\|_{L^2}\to 0 \quad \text{as } t\to\infty. \tag{2.34}$$

The long proof consists of energy estimates using the entropy relation

$$\partial_t(\rho\eta)+\operatorname{div}(\rho u\eta)+\operatorname{div}\left(\frac{q}{\theta}\right)=\frac{q^2}{\kappa\theta^2}+\frac{S_2^2}{\theta\lambda}+\frac{S_1^2}{2\mu\theta} \tag{2.35}$$

for the entropy η defined by

$$\eta := C_v\ln\theta-R\ln\rho+\frac{\tau_1}{2\kappa\theta^2\rho}q^2, \tag{2.36}$$

and the dissipative relation

$$\begin{aligned}
&\partial_t\left[C_v\rho(\theta-\ln\theta-1)+R(\rho\ln\rho-\rho+1)+(1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2+\frac{1}{2}\rho u^2+\frac{\tau_3}{2\lambda}S_2^2\right]\\
&+\operatorname{div}\Big[C_v\rho u(\theta-\ln\theta-1)+u(1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2+\frac{\tau_3}{2\lambda}uS_2^2+R\rho u\ln\rho-R\rho u\\
&-\frac{q}{\theta}+\frac{1}{2}\rho u|u|^2+pu+q-\mu u(\nabla u+\nabla u^T-\frac{2}{n}\operatorname{div}uI_n)-S_2u\Big]\\
&+\frac{q^2}{\kappa\theta^2}+\frac{S_2^2}{\theta\lambda}+\frac{\mu}{2\theta}|\nabla u+\nabla u^T-\frac{2}{n}\operatorname{div}uI_n|^2=0.
\end{aligned} \tag{2.37}$$

Assuming again for simplicity $\tau_1=\tau_3=:\tau$ and letting $(\rho^\tau,u^\tau,\theta^\tau,q^\tau,S_2^\tau)$ denote the local solution defined on $[0,T_\tau)$, we have:

Theorem 2.12 ([33]). *Let (ρ,u,θ) be the smooth solution to the classical compressible Navier-Stokes equations with $(\rho(x,0),u(x,0),\theta(x,0))=(\rho_0,u_0,\theta_0)$ satisfying*

$$\inf_{(x,t)\in\mathbb{R}^3\times[0,T_*]}(\rho(x,t),\theta(x,t))>0$$

and

$$\begin{aligned}
(\rho-1)&\in C([0,T_*],H^6)\cap C^1([0,T_*],H^5),\\
(u,\theta-1)&\in C([0,T_*],H^6)\cap C^1([0,T_*],H^4),
\end{aligned}$$

with finite $T_*>0$. Moreover, assume that the initial data satisfy

$$\|(\rho_0^\tau-\rho_0,u_0^\tau-u_0,\theta_0^\tau-\theta_0,\ \sqrt{\tau}(q_0^\tau+\kappa\nabla\theta_0),\ \sqrt{\tau}(S_{20}^\tau-\lambda\operatorname{div}u_0))\|_{H^3}\le\tau.$$

Then, there exist constants τ_0 and $C > 0$ such that for $\tau \le \tau_0$,

$$\|(\rho^\tau, u^\tau, \theta^\tau)(\cdot, t) - (\rho, u, \theta)(\cdot, t)\|_{H^3} \le C\tau, \tag{2.38}$$

and

$$\|(q^\tau + \kappa\nabla\theta, S_2^\tau - \lambda\nabla u)\|_{H^3} \le C\tau^{\frac{1}{2}}, \tag{2.39}$$

for all $t \in (0, min(T_*, T_\tau))$, and the constant C is independent of τ.

The proof can be done in the spirit of corresponding considerations in [23], overcoming a higher complexity given here by energy estimates similar to those used in the proof of Theorem 2.11.

Peng and Zhao [45] studied the 1D version and obtained in particular a global existence result which is uniform with respect to τ as well as a global convergence result in a weak topology.

For the case $\mu = 0$, we have a local existence theorem. In Section 4, a blow-up result will be presented. The differential equations (1.13) reduce to a purely hyperbolic one with zero-order damping terms,

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \rho\partial_t u + \rho u \cdot \nabla u + \nabla p = \nabla S_2, \\ \rho\partial_t e + \rho u \cdot \nabla e + p\operatorname{div} u + \operatorname{div} q = S_2 \operatorname{div} u, \\ \tau_1(\partial_t q + u \cdot \nabla q) + q + \kappa\nabla\theta = 0, \\ \tau_3(\partial_t S_2 + u \cdot \nabla S_2) + S_2 = \lambda \operatorname{div} u. \end{cases} \tag{2.40}$$

The existence of solutions to (2.40) with initial conditions (2.31), even locally, is not immediately clear, since it is neither symmetric nor strictly hyperbolic. By carefully calculating the eigenvalues and eigenvectors of the corresponding matrix in the associated first-order system, one realizes that it is a *constantly hyperbolic* system, and thus has a local solution.

Assume that there exists $\delta > 0$, sufficiently small, such that

$$\min_{x\in\mathbb{R}^n}(\rho_0(x), \theta_0(x)) > 0, \quad \max_{x\in\mathbb{R}^n}(|\rho_0 - 1|, |\theta_0 - 1|, |q_0(x)|, |S_{20}(x)|) \le \frac{\delta}{2}. \tag{2.41}$$

Then we have:

Theorem 2.13. *([33]) Let $s > \frac{n}{2} + 1$ and $(\rho_0, u_0, \theta_0, q_0, S_{20}) : \mathbb{R}^n \to \mathbb{R}^{2n+3}$ be given with*

$$\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0 \in H^s.$$

Then, there exists a unique local solution $(\rho, u, \theta, q, S_2)$ to system (2.40), (2.31) *in some time interval $[0, T_0)$ with*

$$(\rho - 1, u, \theta - 1, q, S_2) \in C^0([0, T_0), H^s) \cap C^1([0, T_0), H^{s-1})$$

and

$$\min_{(x,t)\in\mathbb{R}^n\times[0,T_0)} (\rho(x,t), \theta(x,t)) > 0,$$

$$\max_{(x,t)\in\mathbb{R}^n\times[0,T_0)} (|\rho(x,t) - 1|, |\theta(x,t) - 1|, |q(x,t)|, |S_2(x,t)|) \le \delta.$$

For the proof, one rewrites the system as a first-order system for $V := (\rho, u, \theta, q, S_2)^T$,

$$\partial_t V + \sum_{j=1}^{3} A_j(V)\partial_{x_j} V + B(V)V = F(V), \tag{2.42}$$

where

$$\sum_{j=1}^{3} A_j\xi_j = \begin{pmatrix} u\cdot\xi & \rho\xi^T & 0 & 0 & 0 \\ \frac{p_\rho}{\rho}\xi & (u\cdot\xi)I_n & \frac{p_\theta}{\rho}\xi & \frac{p_q}{\rho}\xi^T & \frac{p_{S_2}-1}{\rho}\xi \\ 0 & \frac{\theta p_\theta}{\rho e_\theta}\xi^T & (u-\frac{2q}{\rho\theta e_\theta})\xi & \xi^T & 0 \\ 0 & 0 & \frac{\kappa}{\tau_1}\xi & (u\cdot\xi)I_n & 0 \\ 0 & -\frac{\lambda}{\tau_3}\xi^T & 0 & 0 & u\cdot\xi \end{pmatrix},$$

$$B(V) = diag\left\{0, 0, 0, \frac{1}{\tau_1}, \frac{1}{\tau_3}\right\}, F(V) = \left(0, 0, \frac{2}{\kappa\theta}q^2 + \frac{1}{\lambda}S_2^2, 0, 0\right)^T.$$

Since the $(2n+3)\times(2n+3)$-matrix $\sum_{j=1}^{n} A_j\xi_j$ is not symmetric, the system (2.42) is neither symmetric-hyperbolic nor strictly hyperbolic, and a symmetrizer is not obvious. So, the local existence does not follow immediately by the classical theory of symmetric-hyperbolic or strictly hyperbolic systems. Carefully analyzing the dimensions of the eigenspaces of the eigenvalues of the matrix, here using the smallness assumption (2.41), it can be shown that the system is constantly hyperbolic. Referring to [46, Thm. 2.3 and Thm. 10.2], we conclude the local well-posedness.

The last **Model 5**, a modification of Model 3, considers, in one dimension, the differential equations (1.16)–(1.18) together with the constitutive equations (1.19), (1.20), and the initial conditions

$$(\rho(x,0), u(x,0), \theta(x,0), q(x,0), S(x,0)) = (\rho_0, u_0, \theta_0, q_0, S_0). \tag{2.43}$$

It is assumed that, for $\theta > 0$,

$$a(\theta) > 0, a'(\theta) \geq 0, \frac{1}{2}\left(\frac{Z(\theta)}{\theta}\right)' \geq 0 \tag{2.44}$$

holds. The assumption $a'(\theta) \geq 0$ implies $e_\theta \geq C_v > 0$, which make the system (1.16)–(1.18) uniformly hyperbolic without a smallness condition. The third inequality in (2.44) will give the L^2 estimates of q, which will be used in the blow-up result in Section 5. Note that by choosing $Z(\theta) = \frac{\tau_1(\theta)}{\kappa(\theta)} = k\theta^\alpha$ with k as any constant and $1 \leq \alpha < 2$, the assumption (2.44) holds.

Now, we transform the equations (1.16)–(1.18) into a first-order symmetric hyperbolic system. First, we rewrite the equation $(1.16)_3$ for θ as

$$\rho e_\theta\theta_t + \left(\rho u e_\theta - \frac{2a(\theta)}{Z(\theta)}q\right)\theta_x + R\rho\theta u_x + q_x = \frac{2a(\theta)}{\tau_1(\theta)}q^2 + \frac{1}{\mu}S^2. \tag{2.45}$$

Then, we have

$$A^0(U)U_t + A^1(U)U_x + B(U)U = F(U), \tag{2.46}$$

where $U = (\rho, u, \theta, q, S)$ and

$$A^0(U) = \mathrm{diag}\{\tfrac{R\theta}{\rho}, \rho, \tfrac{\rho e_\theta}{\theta}, \tfrac{\tau_1(\theta)\rho}{\kappa(\theta)}, \tfrac{\tau_2\rho}{\mu}\},$$

$$A^1(U) = \begin{pmatrix} \frac{R\theta}{\rho}u & R\theta & 0 & 0 & 0 \\ R\theta & \rho u & R\rho & 0 & -1 \\ 0 & R\rho & \left(\frac{\rho u e_\theta}{\theta} - \frac{2a(\theta)}{\theta Z(\theta)}q\right) & \frac{1}{\theta} & 0 \\ 0 & 0 & \frac{1}{\theta} & \frac{\tau_1(\theta)}{\kappa(\theta)}\rho u & 0 \\ 0 & -1 & 0 & 0 & \frac{\tau_2}{\mu}\rho u \end{pmatrix},$$

$$B(U) = \mathrm{diag}\{0, 0, 0, \tfrac{1}{\kappa\theta}, \tfrac{1}{\mu}\}, F(U) = \mathrm{diag}\{0, 0, -\tfrac{2a(\theta)}{\tau_1(\theta)\theta}q^2 - \tfrac{S^2}{\mu\theta}, 0, 0\}.$$

Therefore, the local existence follows immediately, see [42, 43, 47]:

Theorem 2.14 ([35]). *Let $s \geq 2$. Suppose that*

$$(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0) \in H^s(\mathbb{R})$$

with $\min_x(\rho_0(x), \theta_0(x)) > 0$*, then there exists a unique local solution* (ρ, u, θ, q, S) *to the system* (1.16)–(1.18)*,* (2.43) *in some time interval* $[0, T]$ *with*

$$(\rho - 1, u, \theta - 1, q, S) \in C^0([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R})), \tag{2.47}$$

$$\min_x(\rho(t, x), \theta(t, x)) > 0, \qquad \forall t > 0. \tag{2.48}$$

3. Blow-up in one dimension – I

In this section, we present a blow-up result for *large* data in the one-dimensional case of Model 3. We recall the differential equations (1.8)–(1.10) and the constitutive equations (1.11), (1.12),

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = S_x, \\ \rho e_t + \rho u e_x + p u_x + q_x = S u_x, \end{cases} \tag{3.1}$$

with

$$\tau_1(q_t + u q_x) + q + \kappa\theta_x = 0, \tag{3.2}$$

and

$$\tau_2(S_t + u S_x) + S = \mu u_x, \tag{3.3}$$

$$e = C_v\theta + \frac{\tau_1}{\kappa\theta\rho}q^2 + \frac{\tau_2}{2\mu\rho}S^2, \tag{3.4}$$

$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta}q^2 - \frac{\tau_2}{2\mu}S^2. \tag{3.5}$$

The initial conditions are given by

$$(\rho(x,0), u(x,0), \theta(x,0), S(x,0), q(x,0)) = (\rho_0, u_0, \theta_0, S_0, q_0). \tag{3.6}$$

The local well-posedness and the global existence for *small* data were given in Theorem 2.7 and Theorem 2.8, respectively. When $\tau_1 = \tau_2 = 0$, the system is reduced to the classical compressible Navier-Stokes equations for which smooth solutions exist globally for arbitrary initial data away from a vacuum, see [48]. On the other hand, when the relaxation parameters go to zero, smooth solutions of the system converge to that of the classical system, see Theorem 1.17. This indicates that the relaxed system exhibits a similar qualitative behavior as the classical system. However, and surprisingly, we show that there are in general no C^1-solutions to the system (3.1)–(3.3) for some large initial data. That is, we have another *nonlinear* system where the relaxation process turns a (globally) well-posed system into a not (globally) well-posed one, only visible in the nonlinear system, since the linearized systems behave similarly, see [49]. This sheds light on the difficulties in proving some global existence results in fluid dynamics, and in finding the "correct" model.

We choose $\delta > 0$ small enough such that $p_\rho, p_\theta, e_\theta$ are positive and bounded away from zero and $|p_S|, |p_q|$ are sufficiently small as functions of (ρ, θ, q, S) on

$$\Omega := (1-\delta, 1+\delta) \times (1-\delta, 1+\delta) \times (-\delta, \delta) \times (-\delta, \delta). \tag{3.7}$$

The method to prove the blow-up result is mainly motivated by Sideris' paper [41] where he showed that any C^1 solutions of compressible Euler equations must blow up in finite time. As was shown in [19], the system (3.1)–(3.3) is a strictly hyperbolic system which implies the property of finite propagation speed, which in turn allows one to define some averaged quantities as in [41] and finally show a blow-up of solutions in finite time by establishing a Riccati-type inequality.

The finite propagation speed is expressed in:

Lemma 3.1. *([49]) Let (ρ, u, θ, q, S) be a local solution to* (3.1)–(3.3), (3.6) *on* $[0, T_0)$. *Let $M > 0$. Assume that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)$ are compactly supported in $(-M, M)$ and $(\rho_0, \theta_0, q_0, S_0) \in \Omega$. Then, there exists a constant σ such that*

$$(\rho(\cdot,t), u(\cdot,t), \theta(\cdot,t), q(\cdot,t), S(\cdot,t)) = (1,0,1,0,0) := (\bar\rho, \bar u, \bar\theta, \bar q, \bar S)$$

on $D(t) = \{x \in \mathbb{R} \big| |x| \geq M + \sigma t\}, \quad 0 \leq t < T_0.$

The averaged quantities used are

$$F(t) := \int_{\mathbb{R}} x\rho(x,t)u(x,t)\mathrm{d}x, \tag{3.8}$$

$$G(t) := \int_{\mathbb{R}} (\mathcal{E}(x,t) - \bar{\mathcal{E}})\mathrm{d}x, \tag{3.9}$$

where

$$\mathcal{E}(x,t) := \rho(e + \frac{1}{2}u^2)$$

is the total energy and

$$\bar{\mathcal{E}} := \bar\rho(\bar e + \frac{1}{2}\bar u^2) = C_v.$$

Now the blow-up result is given by:

Theorem 3.2. *([49]) Assuming*

$$G(0) > 0, \tag{3.10}$$

there exists u_0 satisfying

$$F(0) > \max\left\{\frac{32\sigma \max\rho_0}{3-\gamma}, \frac{4\sqrt{\max\rho_0}}{\sqrt{3-\gamma}}\right\} M^2, \quad 1 < \gamma := 1 + \frac{R}{C_v} < 3 \tag{3.11}$$

such that the length T_0 of the maximal interval of existence of a smooth solution (ρ, u, θ, q, S) to (3.1)–(3.3)*, 3.6 is finite, provided the compact support of the initial data is sufficiently large.*

The assumption $1 < \gamma < 3$ holds for the elementary kinetic theory of gases, cf. [41]. Note that it is assumed that the local solution satisfies $(\rho, \theta, q, S)(t) \in \Omega$. This a priori assumption does not affect u, which blows up, but is a restriction; the solutions might reach the boundary. In Section 5, we will consider Model 5, a modification of Model 3, and show a blow-up result excluding this possibility.

Sketch of the proof of Theorem 3.2: Using the constitutive equations and the constancy of G, we get

$$\begin{aligned} F'(t) \quad \geq \quad & \frac{3-\gamma}{2}\int_{\mathbb{R}} \rho u^2 \mathrm{d}x - \int_{\mathbb{R}} \frac{\tau_1(2\gamma-1)}{2\kappa\theta} q^2 \mathrm{d}x - \int_{\mathbb{R}} (\frac{\tau_2(2\gamma-1)}{2\mu} + \frac{1}{2}) S^2 \mathrm{d}x \\ & -(M+\sigma t). \end{aligned} \tag{3.12}$$

On the other hand

$$\begin{aligned} F^2(t) &= \left(\int_{\mathbb{R}} x\rho(x,t)u(x,t)\mathrm{d}x\right)^2 \\ &\leq \int_{B_t} x^2\rho \mathrm{d}x \cdot \int_{B_t} \rho u^2 \mathrm{d}x \\ &\leq (M+\tilde{\sigma}t)^2 \int_{B_t} \rho \mathrm{d}x \cdot \int_{B_t} \rho u^2 \mathrm{d}x \\ &= (M+\tilde{\sigma}t)^2 \int_{B_t} \rho_0 \mathrm{d}x \cdot \int_{B_t} \rho u^2 \mathrm{d}x \\ &\leq 2\max\rho_0 (M+\tilde{\sigma}t)^3 \int_{\mathbb{R}} \rho u^2 \mathrm{d}x, \end{aligned}$$

where $B_t = \{x \in \mathbb{R} \mid |x| \leq M + \tilde{\sigma}t\} = (-(M+\tilde{\sigma}t), M+\tilde{\sigma}t)$ and $\tilde{\sigma} \geq \sigma$ can be chosen arbitrary. For simplicity, we still denote $\tilde{\sigma}$ by σ in the following calculations. Therefore, we have

$$\begin{aligned} F'(t) \quad \geq \quad & \frac{3-\gamma}{4\max\rho_0(M+\sigma t)^3} F^2 - \int_{\mathbb{R}} \frac{\tau_1(2\gamma-1)}{2\kappa\theta} q^2 \mathrm{d}x - \int_{\mathbb{R}} \frac{\tau_2(2\gamma-1)+\mu}{2\mu} S^2 \mathrm{d}x \\ & -(M+\sigma t). \end{aligned} \tag{3.13}$$

Let

$$c_2 := \frac{\sigma}{M}, \qquad c_3 := \frac{3-\gamma}{4\max\rho_0 M^3}.$$

Assume, for the moment a priori,

$$F(t) \geq c_1 > 0 \tag{3.14}$$

and

$$M + \sigma t = M(1 + c_2 t) \leq \frac{c_3}{2(1 + c_2 t)^3} F^2, \tag{3.15}$$

where c_1 is to be determined later. Then

$$\frac{F'(t)}{F^2} \geq \frac{c_3}{2(1 + c_2 t)^3} - \frac{\tau_1(2\gamma - 1)}{c_1^2 \kappa \bar{\theta}} \int_{\mathbb{R}} q^2 \mathrm{d}x - \frac{\tau_2(2\gamma - 1) + \mu}{c_1^2 2\mu} \int_{\mathbb{R}} S^2 \mathrm{d}x. \tag{3.16}$$

Using the identity (2.27) and defining

$$H_0 := \int_{\mathbb{R}} \left(C_v \rho_0(\theta_0 - \ln \theta_0 - 1) + R(\rho_0 \ln \rho_0 - \rho_0 + 1) + (1 - \frac{1}{2\theta_0}) \frac{\tau_1}{\kappa \theta} q_0^2 + \frac{\tau_2}{2\mu} S_0^2 \right) \mathrm{d}x,$$

we obtain

$$\frac{\tau_1(2\gamma - 1)}{c_1^2 \kappa \bar{\theta}} \int_0^t \int_{\mathbb{R}} q^2 \mathrm{d}x\mathrm{d}t + \frac{\tau_2(2\gamma - 1) + \mu}{c_1^2 2\mu} \int_0^t \int_{\mathbb{R}} S^2 \mathrm{d}x\mathrm{d}t \leq c_4 + c_5 \|u_0\|_{L^2}^2, \tag{3.17}$$

where

$$c_4 := \frac{1}{c_1^2} \left[\bar{\theta}(4\tau_1(2\gamma - 1) + \tau_2(2\gamma - 1) + \mu) H_0 \right]$$

and

$$c_5 := \frac{1}{c_1^2} \left[\bar{\theta}(4\tau_1(2\gamma - 1) + \tau_2(2\gamma - 1) + \mu) \frac{\max \rho_0}{2} \right].$$

Integrating the inequality (3.16), we get

$$\frac{1}{F_0} - \frac{1}{F} \geq -\frac{c_3}{4c_2(1 + c_2 t)^2} + \frac{c_3}{4c_2} - c_4 - c_5 \|u_0\|_{L^2}^2. \tag{3.18}$$

Now we assume additionally and a priori

$$F_0 > \frac{8c_2}{c_3} \tag{3.19}$$

and

$$c_4 + c_5 \|u_0\|_{L^2}^2 \leq \frac{c_3}{8c_2}. \tag{3.20}$$

Then, we get

$$\frac{1}{F_0} \geq \frac{1}{F_0} - \frac{1}{F} \geq -\frac{c_3}{4c_2(1 + c_2 t)^2} + \frac{c_3}{8c_2}, \tag{3.21}$$

which means that T_0 cannot be arbitrarily large without contradicting (3.19). It remains to show that the a priori assumptions (3.14), (3.15), (3.19), and (3.20) can be justified.

(3.14) is easy to show with $c_1 := \frac{2c_2}{c_3}$. For (3.15) to hold, we only need to show the following inequality:

$$M(1 + c_2 t) \leq \frac{c_3}{4(1 + c_2 t)^3} F^2. \tag{3.22}$$

For $t = 0$, it is sufficient to guarantee

$$\sigma^2 \geq \frac{3-\gamma}{16\max\rho_0}, \tag{3.23}$$

which is satisfied naturally since σ can be chosen arbitrarily large. Thus, the proof will be finished if we can show the existence of u_0 such that (3.19) and (3.20) hold, and the assumption (3.10) is satisfied. As in [19], we choose $u_0 \in H^2(\mathbb{R}) \cap C^1(\mathbb{R})$ as follows:

$$u_0(x) := \begin{cases} 0, & x \in (-\infty, -M], \\ \frac{L}{2}\cos(\pi(x+M)) - \frac{L}{2}, & x \in (-M, -M+1], \\ -L, & x \in (-M+1, -1], \\ L\cos(\frac{\pi}{2}(x-1)), & x \in (-1, 1], \\ L, & x \in (1, M-1], \\ \frac{L}{2}\cos(\pi(x-M+1)) + \frac{L}{2}, & x \in (M-1, M], \\ 0, & x \in (M, \infty), \end{cases} \tag{3.24}$$

where L is a positive constant to be determined later. We assume $M \geq 4$. Assumption (3.10) can easily be satisfied since it is equivalent to requiring

$$\int_{\mathbb{R}} \left(\rho_0 e_0 - \bar{\rho}\bar{e} + \frac{1}{2}u_0^2\right) dx > 0,$$

which is satisfied by choosing $\rho_0\theta_0 > \bar{\rho}\bar{\theta} = 1$. Since

$$F_0 = \int_{\mathbb{R}} x\rho_0(x)u_0(x)\mathrm{d}x \geq \frac{L}{2}\min\rho_0 M^2,$$

we can choose L large enough, independent of M, such that

$$\frac{L}{2}\min\rho_0 > \max\left\{\frac{32\sigma\max\rho_0}{3-\gamma}, \frac{4\sqrt{\max\rho_0}}{\sqrt{3-\gamma}}\right\}$$

implying (3.19). On the other hand, since $\|u_0\|_{L^2}^2 \leq 2L^2M$, we can choose M sufficiently large such that

$$\bar{\theta}(8\gamma\tau_1 + 2\gamma\tau_2 + \mu)(H_0 + \max\rho_0 ML^2) \leq \frac{2\sigma\max\rho_0}{(3-\gamma)}M^2$$

holds, implying (3.20), and the proof is finished.

4. Blow-up in multi-dimensions

A blow-up result is presented for Model 4 in the case $\mu = 0$, in dimensions $n = 2, 3$. We recall the differential equations (2.40) and the initial conditions (2.31):

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \rho\partial_t u + \rho u\cdot\nabla u + \nabla p = \nabla S_2, \\ \rho\partial_t e + \rho u\cdot\nabla e + p\mathrm{div}u + \mathrm{div}q = S_2\mathrm{div}u, \\ \tau_1(\partial_t q + u\cdot\nabla q) + q + \kappa\nabla\theta = 0, \\ \tau_3(\partial_t S_2 + u\cdot\nabla S_2) + S_2 = \lambda\,\mathrm{div}u, \end{cases} \tag{4.1}$$

$$(\rho(x,0), u(x,0), \theta(x,0), q(x,0), S_2(x,0)) = (\rho_0, u_0, \theta_0, q_0, S_{20}). \tag{4.2}$$

Additionally, we assume the specified constitutive equations (1.14), (1.15),

$$e = C_v\theta + \frac{\tau_1}{\kappa\rho\theta}q^2 + \frac{\tau_3}{2\lambda\rho}S_2^2, \tag{4.3}$$

$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta}q^2 - \frac{\tau_3}{2\lambda}S_2^2. \tag{4.4}$$

A local solution was given in Theorem 2.13. For the blow-up result, we need to assume there exists $\delta > 0$, sufficiently small, such that

$$\max_{x\in\mathbb{R}^n}(|\rho_0 - 1|, |\theta_0 - 1|, |q_0(x)|, |S_{20}(x)|) \leq \frac{\delta}{2} \tag{4.5}$$

and that this implies on the interval of local existence

$$\max_{x\in\mathbb{R}^n}(|\rho_0 - 1|, |\theta_0 - 1|, |q_0(x)|, |S_{20}(x)|) \leq \delta. \tag{4.6}$$

We remark that this assumption does not affect u which is shown to blow-up in finite time. The finite propagation speed of the hyperbolic system is expressed in:

Lemma 4.1. *([33]) Let $(\rho, u, \theta, q, S_2)$ be the local solution to* (4.1), (4.2) *on* $[0, T_0)$. *We further assume that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20})$ are compactly supported in a ball $B_0(M)$ with radius $M > 0$. Then, there exists a constant σ such that*

$$(\rho(\cdot,t), u(\cdot,t), \theta(\cdot,t), q(\cdot,t), S_2(\cdot,t)) = (1,0,1,0,0) =: (\bar\rho, \bar u, \bar\theta, \bar q, \bar S_2) \tag{4.7}$$

on $D(t) := \{x \in \mathbb{R}^n \| x| \geq M + \sigma t\}$, $0 \leq t < T_0$.

The following averaged quantities are used, cf. Section 3:

$$F(t) := \int_{\mathbb{R}^n} x \cdot \rho(x,t)u(x,t)\mathrm{d}x, \tag{4.8}$$

$$G(t) := \int_{\mathbb{R}^n} (\mathcal{E}(x,t) - \bar{\mathcal{E}})\mathrm{d}x, \tag{4.9}$$

where $\mathcal{E}(x,t) := \rho(e + \frac{1}{2}u^2)$ is the total energy and $\bar{\mathcal{E}} := \bar\rho(\bar e + \frac{1}{2}\bar u^2) = C_v$. Then we have:

Theorem 4.2. *Let $(\rho, u, \theta, q, S_2)$ be the local solution to* (4.1), (4.2) *on* $[0, T_0)$. *Assume that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20})$ are compactly supported in a ball $B_0(M)$ with radius $M > 0$. Moreover, we assume that*

$$G(0) > 0, \tag{4.10}$$

$$1 < \gamma := 1 + \frac{R}{C_v} < \frac{5}{3}. \tag{4.11}$$

Then, there exists u_0 satisfying

$$F(0) > \max\left\{\frac{128\sigma \max\rho_0}{3(5-3\gamma)}, \frac{8\sqrt{\pi}\max\rho_0}{\sqrt{3(5-3\gamma)}}\right\} M^4, \tag{4.12}$$

such that the length T_0 of the maximal interval of existence of a smooth solution $(\rho, u, \theta, q, S_2)$ is finite, provided the compact support of the initial data is sufficiently large.

This blow-up result relies on assumption (4.6) assuring the remaining of the solutions in the hyperbolic region, cf. the remarks in Section 3. A modification of the system, as done for Model 3 in Model 5, might remove this assumption, see Sections 3 and 5. The proof is similar to that of Theorem 3.2 in Section 3 and is presented for the case $n = 3$. The case $n = 2$ is proved by easy modifications.

Sketch of the proof of Theorem 4.2: Using the constancy of G and the constitutive equations (4.3), (4.4), we conclude

$$F'(t) \geq \frac{5-3\gamma}{2}\int_{\mathbb{R}^3}\rho u^2 \mathrm{d}x - 3\int_{\mathbb{R}^3}\frac{\tau_1\gamma}{\kappa\theta}q^2\mathrm{d}x - 3\int_{\mathbb{R}^3}\left(\frac{\tau_3\gamma}{\lambda}+\frac{1}{2}\right)S_2^2\mathrm{d}x - 2\pi(M+\sigma t)^3. \tag{4.13}$$

Similar to the one-dimensional case discussed in Section 3, we obtain

$$F'(t) \geq \frac{3(5-3\gamma)}{8\pi\max\rho_0(M+\sigma t)^5}F^2 - 3\int_{\mathbb{R}^3}\left(\frac{\tau_1\gamma}{\kappa\theta}q^2 + \frac{2\tau_3\gamma+\lambda}{2\lambda}S_2^2\right)\mathrm{d}x - 2\pi(M+\sigma t)^3. \tag{4.14}$$

Let $c_2 := \frac{\sigma}{M}$, $c_3 := \frac{3(5-3\gamma)}{8\pi\max\rho_0 M^5}$. We assume a priori for the moment

$$F(t) \geq c_1 > 0 \tag{4.15}$$

and

$$2\pi(M+\sigma t)^3 = 2\pi M^3(1+c_2t)^3 \leq \frac{c_3}{2(1+c_2t)^5}F^2, \tag{4.16}$$

where c_1 is to be determined later. Then

$$\frac{F'(t)}{F^2(t)} \geq \frac{c_3}{2(1+c_2t)^5} - \frac{6\tau_1\gamma}{c_1^2\kappa\bar{\theta}}\int_{\mathbb{R}^3}q^2\mathrm{d}x - \frac{6\tau_3\gamma+3\lambda}{c_1^2 2\lambda}\int_{\mathbb{R}^3}S_2^2\mathrm{d}x. \tag{4.17}$$

Using the the dissipative entropy equation (2.37), with $\mu = 0$, and

$$\begin{aligned} W_0 \;:=\; & \int_{\mathbb{R}^3}\Big(C_v\rho_0(\theta_0 - \ln\theta_0 - 1) + R(\rho_0\ln\rho_0 - \rho_0 + 1) + \\ & (1-\frac{1}{2\theta_0})\frac{\tau_1}{\kappa\theta}q_0^2 + \frac{\tau_2}{2\lambda}S_{20}^2\Big)\mathrm{d}x, \end{aligned}$$

we obtain

$$\frac{6\tau_1\gamma}{c_1^2\kappa\bar{\theta}}\int_0^t\int_{\mathbb{R}^3}q^2\mathrm{d}x\mathrm{d}t + \frac{6\tau_3\gamma+3\lambda}{c_1^2 2\lambda}\int_0^t\int_{\mathbb{R}^3}S_2^2\mathrm{d}x\mathrm{d}t \leq c_4 + c_5\|u_0\|_{L^2}^2, \tag{4.18}$$

where

$$c_4 = \frac{3}{c_1^2}\left[\bar{\theta}(8\tau_1\gamma + 2\tau_3\gamma + \lambda)W_0\right], \quad c_5 = \frac{3}{c_1^2}\left[\bar{\theta}(8\tau_1\gamma + 2\tau_3\gamma+\lambda)\frac{\max\rho_0}{2}\right].$$

Integrating (4.17), we have

$$\frac{1}{F_0} - \frac{1}{F(t)} \geq -\frac{c_3}{8c_2(1+c_2t)^4} + \frac{c_3}{8c_2} - c_4 - c_5\|u_0\|_{L^2}^2. \tag{4.19}$$

Now, we additionally assume a priori

$$F_0 > \frac{16c_2}{c_3}, \tag{4.20}$$

$$c_4 + c_5\|u_0\|_{L^2}^2 \le \frac{c_3}{16c_2}. \tag{4.21}$$

Then we get

$$\frac{1}{F_0} \ge \frac{1}{F_0} - \frac{1}{F(t)} \ge -\frac{c_3}{8c_2(1+c_2t)^4} + \frac{c_3}{16c_2} \tag{4.22}$$

which implies that the maximal time of existence T cannot be arbitrarily large without contradicting (4.20). It remains to show that the a priori assumptions (4.15), (4.16), (4.20), and (4.21) can be justified.

(4.15) is easy to show with $c_1 := \frac{4c_2}{c_3}$. For (4.16) to hold, it suffices to show

$$2\pi M^3(1+c_2t)^3 \le \frac{c_3}{4(1+c_2t)^5}F(t)^2. \tag{4.23}$$

For $t = 0$, it is sufficient to guarantee

$$\sigma^2 \ge \frac{3(5-3\gamma)}{64\max\rho_0}, \tag{4.24}$$

which is satisfied naturally since σ can be chosen arbitrarily large. Thus, the proof will be finished if we can show the existence of u_0 such that (4.20) and (4.21) hold and the assumption (4.10) is satisfied. Let

$$\tilde{v}(r) = \begin{cases} L\cos(\frac{\pi}{2}(r-1)), & r \in [0,1], \\ L, & r \in (1, M-1], \\ \frac{L}{2}\cos(\pi(r-M+1)) + \frac{L}{2}, & r \in (M-1, M], \\ 0, & r \in (M, +\infty), \end{cases} \tag{4.25}$$

where L is a positive constant to be determined later. $\tilde{v}$ is not in $H^3(\mathbb{R}_+)$, but we can think of $\tilde{v}$ being smoothed around the singular points $r = 1, M-1, M$ and put to zero around $r = 0$, yielding a function v, with $\|v\|_{L^2} \le 2\|\tilde{v}\|_{L^2}$. We choose

$$u_0(x) := v(|x|)\frac{x}{|x|}.$$

Assumption (4.10) can easily be satisfied since it is equivalent to requiring

$$\int_{\mathbb{R}^3}\left(\rho_0 e_0 - \bar{\rho}\bar{e} + \frac{1}{2}u_0^2\right)dx > 0,$$

which is satisfied by choosing $\rho_0\theta_0 > \bar{\rho}\bar{\theta} = 1$. Let $M \ge 5$. Since

$$F_0 = \int_{\mathbb{R}^3} x\cdot\rho_0(x)u_0(x)\mathrm{d}x \ge \frac{\pi\min\rho_0}{32}LM^4,$$

we can choose L sufficiently large, independent of M, such that

$$\frac{\pi\min\rho_0}{32}L \ge \max\left\{\sqrt{\frac{64\pi\max\rho_0}{3(5-3\gamma)}}, \frac{128\sigma\pi\max\rho_0}{3(5-3\gamma)}\right\},$$

implying (4.20). On the other hand, since $\|u_0\|_{L^2}^2 \le 4L^2\frac{4\pi}{3}M^3$, we can choose M sufficiently large such that

$$\bar{\theta}(8\tau_1\gamma + 2\tau_2\gamma + \mu)\left(W_0 + \frac{2\pi \max\rho_0 L^2}{3}M^3\right) \le \frac{16\pi\sigma \max\rho_0}{9(5-3\gamma)}M^4$$

holds, implying (4.21), and the proof is finished.

5. Blow-up in one dimension – II

Here we present a second blow-up result in one dimension, for Model 5 being a modification of Model 3, avoiding the possibility of reaching the hyperbolic boundary, cf. Section 3. We recall the differential equations and the initial conditions,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = S_x, \\ E_t + (uE + pu + q - Su)_x = 0, \end{cases} \tag{5.1}$$

where E represents the total energy,

$$\tau_1(\theta)(\rho q_t + \rho u \cdot q_x) + q + \kappa(\theta)\theta_x = 0, \tag{5.2}$$

and

$$\tau_2(\rho S_t + \rho u \cdot S_x) + S = \mu u_x, \tag{5.3}$$

$$(\rho(x,0), u(x,0), \theta(x,0), q(x,0), S(x,0)) = (\rho_0, u_0, \theta_0, q_0, S_0), \tag{5.4}$$

as well as the constitutive equations

$$E = \frac{1}{2}\rho u^2 + \frac{\tau_2}{2\mu}\rho S^2 + \rho e(\theta, q), \tag{5.5}$$

and the specific internal energy e and the pressure p are given by

$$e(\theta) = C_v\theta + a(\theta)q^2, \qquad p(\rho,\theta) = R\rho\theta, \tag{5.6}$$

where

$$a(\theta) = \frac{Z(\theta)}{\theta} - \frac{1}{2}Z'(\theta) \quad \text{with} \quad Z(\theta) = \frac{\tau_1(\theta)}{\kappa(\theta)}.$$

The local existence is given in Theorem 2.14. Neglecting ρ in the constitutive relations (1.3)–(1.5) and assuming τ_1, κ to be constants, in Section 3 a blow-up result was established under the assumption that $(\rho-1, \theta-1, q, S) \in \Omega$ with $\Omega = ((-\delta,\delta))^4$ being a "small" domain requiring δ to be sufficiently small to assure that the arising system is — though non-symmetric — strictly hyperbolic, which, in turn, assures the local solvability. This smallness of $|(\rho-1, \theta-1, q, S)|$ — notice: not including u — has been established globally only for small data. Therefore, the solutions in Section 3 might "blow up" in the sense that one may reach the boundary of Ω. In the present paper, the system is a *symmetric* hyperbolic one, not requiring any smallness condition of this kind.

The most interesting aspect, as in Section 3, might be that the blow-up result contrasts the situation without relaxation. i.e., for the classical compressible Navier-Stokes system corresponding to $\tau_1 = \tau_2 = 0$, where large global solutions exist, see Kazhikhov [48]. This really nonlinear effect — loosing the global existence for large data —, not anticipated from the linearized version, shows the possible impact a relaxation might have. For several linear systems of various type an effect is visible in loosing exponential stability in bounded domains or becoming of regularity loss type in the Cauchy problem, see the discussion in our paper [49].

The method we use to prove the blow-up result is mainly motivated by Sideris' paper [41] where he showed that any C^1-solutions of compressible Euler equations must blow up in finite time. A blow-up result for a similar system has also been proved recently by Freistühler [50] applying the general result for hyperbolic systems with sources in one space dimension by Bärlin [51]. A solution remains bounded, but the solution does not remain in C^1, provided the data are *small* enough. In contrast to [50, 51], our blow-up requires *large* initial velocities; moreover, here the largeness is described explicitly. For initial data being small in higher-order Sobolev spaces (H^2), there exist global solutions. The method used here, and before in Sections 3 and 4, also extends to higher dimensions, as seen in Section 4.

Since the system is symmetric hyperbolic, the local solution possesses the finite propagation speed property:

Lemma 5.1. *Let (ρ, u, θ, q, S) be the local solution according to Theorem 2.14 on $[0, T_0)$. Let $M > 0$. We assume that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)$ are compactly supported in $(-M, M)$. Then, there exists a constant σ such that*

$$(\rho(\cdot,t), u(\cdot,t), \theta(\cdot,t), q(\cdot,t), S(\cdot,t) = (1,0,1,0,0) := (\bar\rho, \bar u, \bar\theta, \bar q, \bar S)$$

on $D(t) = \{x \in \mathbb{R} \big| |x| \geq M + \sigma t\}, \quad 0 \leq t < T_0.$

We define again some averaged quantities,

$$F(t) := \int \rho u \cdot x \mathrm{d}x - \tau_2 \int \rho S \,\mathrm{d}x, \tag{5.7}$$

$$G(t) := \int_{\mathbb{R}} (E(x,t) - \bar E) \mathrm{d}x, \tag{5.8}$$

where

$$E = \frac{1}{2}\rho u^2 + \frac{\tau_2}{2\mu}\rho S^2 + \rho e(\theta, q)$$

is the total energy and

$$\bar E := \bar\rho(\bar e + \frac{1}{2}\bar u^2) = C_v.$$

The functional F with the second term involving S is different from those used in [41] and [49] (resp., Section 3). This second term is new and technically motivated. The blow-up result is now given by:

Theorem 5.2. *We assume*

$$G(0) > 0. \tag{5.9}$$

Then, there exists $(\rho_0, u_0, \theta_0, q_0, S_0)$ *satisfying*

$$F(0) > \frac{32\sigma \max\rho_0}{3-\gamma} M^2 \tag{5.10}$$

and

$$4\left(\frac{(3-\gamma)\mu\tau_2}{M^2} + \gamma - 1\right)(H_0 + \frac{\max\rho_0}{2}\|u_0\|_{L^2}^2) \leq \frac{128\sigma^2 \max\rho_0 M}{3-\gamma}, \tag{5.11}$$

where

$$\begin{aligned} H_0 \;\; := \;\; & \int C_v\rho_0(\theta_0 - \ln\theta_0 - 1) + R(\rho_0 \ln\rho_0 - \rho_0 + 1) + \\ & \rho_0\left(a(\theta_0) + \frac{1}{2}\left(\frac{Z(\theta_0)}{\theta_0}\right)'\right) q_0^2 + \frac{\tau_2}{2\mu} S_0^2 \,\mathrm{d}x, \end{aligned} \tag{5.12}$$

such that the length T_0 *of the maximal interval of existence of a smooth solution* (ρ, u, θ, q, S) *to system* (1.16)–(1.18), (2.43) *is finite, provided the compact support of the initial data is sufficiently large and* $\gamma := 1 + \frac{R}{C_v}$ *is sufficiently close to* 1.

Sketch of the proof: It is in the line of the proofs of the blow-up theorems in Sections 3 and 4, but with a slightly higher complexity due to the necessarily modified quantity F, but also with an improved strategy.

The entropy η, defined by

$$\eta := C_v \ln\theta - R\ln\rho - \left(\frac{Z(\theta)}{2\theta}\right)' q^2, \tag{5.13}$$

satisfies

$$(\rho\eta)_t + \left(\rho u\eta + \frac{q}{\theta}\right)_x = \frac{q^2}{\kappa(\theta)\theta^2} + \frac{S^2}{\mu\theta}. \tag{5.14}$$

Using this and the constancy of G, we can derive

$$F'(t) \geq \frac{3-\gamma}{2}\int \rho u^2 \mathrm{d}x - (\gamma-1)(H_0 + \frac{\max\rho_0}{2}\|u_0\|_{L^2}^2).$$

On the other hand, F satisfies

$$\int \rho u^2 \mathrm{d}x \geq \frac{F(t)^2}{4\max\rho_0(M+\sigma t)^3} - \frac{2\mu\tau_2(H_0 + \frac{\max\rho_0}{2}\|u_0\|_{L^2}^2)}{(M+\sigma t)^2}.$$

The last two estimates imply

$$\begin{aligned} F'(t) &\geq \frac{3-\gamma}{8\max\rho_0(M+\sigma t)^3} F^2(t) - \left(\frac{(3-\gamma)\mu\tau_2}{(M+\sigma t)^2} + \gamma - 1\right)(H_0 + \frac{\max\rho_0}{2}\|u_0\|_{L^2}^2) \\ &\equiv \frac{c_3}{(1+c_2 t)^3} F(t)^2 - K(t) \end{aligned} \tag{5.15}$$

where $c_2 := \frac{\sigma}{M}, c_3 := \frac{3-\gamma}{8 \max \rho_0 M^3}$. With this Riccati-type inequality, we can show the blow-up result. Indeed, assuming a priori that

$$2K(t) \leq \frac{c_3}{(1+c_2 t)^3} F^2(t), \tag{5.16}$$

we have

$$F'(t) \geq \frac{c_3}{2(1+c_2 t)^3} F^2(t),$$

which gives

$$\frac{1}{F(0)} \geq \frac{1}{F(0)} - \frac{1}{F(t)} \geq \frac{c_3}{4c_2} - \frac{c_3}{4c_2(1+c_2 t)^2}. \tag{5.17}$$

Hence, the maximal existence time T_0 cannot be infinite provided

$$F(0) > \frac{4c_2}{c_3} = \frac{32\sigma \max \rho_0 M^2}{3-\gamma}, \tag{5.18}$$

which is equivalent to assumption (5.10). Using (5.18), we get

$$\frac{1}{F(t)} \leq \frac{1}{F(0)} - \frac{c_3}{4c_2} + \frac{c_3}{4c_2(1+c_2 t)^2} \leq \frac{c_3}{4c_2(1+c_2 t)^2}, \tag{5.19}$$

which implies

$$F(t) \geq \frac{4c_2(1+c_2 t)^2}{c_3}. \tag{5.20}$$

To show that the a priori estimate (5.16) holds, we use the bootstrap method expressed in the following simple lemma.

Lemma 5.3. *Let* $f \in C^0([0,\infty),[0,\infty))$ *and* $0 < a < b$ *such that the following holds for any* $0 \leq \alpha < \beta < \infty$*:*

$$f(0) < a \qquad \text{and} \qquad (\forall\, t \in [\alpha,\beta] : f(t) \leq b \implies \forall\, t \in [\alpha,\beta] : f(t) \leq a.).$$

Then we have

$$\forall t \geq 0 : \; f(t) \leq a.$$

We will apply this lemma in the time domain of existence to f, a, b with

$$f(t) := \frac{K(t)(1+c_2 t)^3}{F^2(t) c_3}, \quad a := \frac{1}{4}, \quad b := \frac{1}{2}.$$

That is, we need to show that

$$4K(t) \leq \frac{c_3}{(1+c_2 t)^3} F^2(t). \tag{5.21}$$

Next, to get (5.21), using (5.20), one only needs to show

$$4K(t)\frac{(1+c_2t)^3}{c_3} \le \frac{16c_2^2}{c_3^2}(1+c_2t)^4 \tag{5.22}$$

for which it is sufficient to show

$$4\left(\frac{(3-\gamma)\mu\tau_2}{M^2}+\gamma-1\right)(H_0+\frac{\max\rho_0}{2}\|u_0\|_{L^2}^2) \le \frac{16c_2^2}{c_3}, \tag{5.23}$$

since K is a decreasing function. The last inequality is equivalent to assumption (5.11). This proves (5.16).

Finally, we need to find some u_0 such that the assumptions (5.10) and (5.11) hold. We choose, similarly to Sections 3 and 4, $u_0 \in H^2(\mathbb{R}) \cap C^1(\mathbb{R})$ as follows:

$$u_0(x) := \begin{cases} 0, & x \in (-\infty, -M], \\ \frac{L}{2}\cos(\pi(x+M)) - \frac{L}{2}, & x \in (-M, -M+1], \\ -L, & x \in (-M+1, -1], \\ L\cos(\frac{\pi}{2}(x-1)), & x \in (-1, 1], \\ L, & x \in (1, M-1], \\ \frac{L}{2}\cos(\pi(x-M+1)) + \frac{L}{2}, & x \in (M-1, M], \\ 0, & x \in (M, \infty), \end{cases} \tag{5.24}$$

where L is a positive constant to be determined later. We assume $M \ge 4$. Assumption (5.9) can easily be satisfied since it is equivalent to requiring

$$\int_{\mathbb{R}}\left(\rho_0 e_0 - \bar\rho\bar e + \frac{1}{2}u_0^2\right)dx > 0,$$

which is satisfied by choosing $\rho_0\theta_0 > \bar\rho\bar\theta = 1$. Since

$$\int_{\mathbb{R}}(x\rho_0(x)u_0(x))\mathrm{d}x \ge \frac{L}{2}\min\rho_0 M^2$$

and

$$\left|\tau_2\int\rho_0 S_0\mathrm{d}x\right| \le \int_{-M}^{M}\rho_0\mathrm{d}x + \tau_2\int\rho_0 S_0^2\mathrm{d}x \le \max\rho_0(1+\mu H_0^2)M^2,$$

we choose L large enough, and independent of M, such that

$$\frac{L}{4}\min\rho_0 > \max\{\max\rho_0(1+\mu H_0^2), \frac{32\sigma\max\rho_0}{3-\gamma}\}.$$

Therefore, (5.10) holds. Now, after having chosen σ large enough, fix L. Then we choose M sufficiently large and $\gamma-1$ sufficiently small such that (5.11) holds. This finishes the proof.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author acknowledges with great gratitude the inspiring mathematical environment and hospitality experienced during various stays with Professor Thomas C. Sideris at the University of California at Santa Barbara.

The successful co-operation with Professor Yuxi Hu leading to the results presented in this survey is also gratefully acknowledged.

Conflict of interest

The author declares no conflict of interest in this paper.

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