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Research article

Nonlinear Schrödinger equation with a short-range compensating field

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Abstract: Within the self-consistent Maxwell-Pauli theory, a nonlinear Schrödinger equation with a short-range compensating field was derived. Stationary and nonstationary solutions of the obtained nonlinear Schrödinger equation for the hydrogen atom were investigated. It is shown that spontaneous emission and the associated rearrangement of the internal structure of the atom, which is traditionally called a spontaneous transition, have a simple and natural description within the classical field theory without any quantization and additional hypotheses. The solution of the nonlinear Schrödinger equation shows that, depending on the frequency of spontaneous emission, the compensating field behaves differently. At relatively low frequencies of spontaneous emission, there is no radiation (waves) of the short-range compensating field, and this field does not carry away energy. In this case, the damping rate of the spontaneous emission coincides with that obtained in quantum electrodynamics (QED). At relatively high frequencies of spontaneous emission, radiation (waves) of the compensating field arises, which, along with electromagnetic radiation, carry away some of the energy. In this case, the damping rate of the spontaneous emission is greater (up 1.5 times) than that predicted by QED.

Keywords: classical field theory; self-consistent Maxwell-Pauli theory; nonlinear Pauli equation; nonlinear Schrödinger equation; spontaneous emission
Mathematics Subject Classification: 81Sxx, 81P10, 81Q80

1. Introduction

At present, it is generally accepted that a complete description of processes in the microworld is given by quantum electrodynamics (QED), in which both the atom (electron) and the

electromagnetic field are quantized. Along with this, the so-called semi-classical theories [1−8], neoclassical theory [9−17] and purely classical field theory [18−23] are successfully used to describe various quantum processes, in which the atom is considered quantum mechanically, i.e., based on wave equations (Schrödinger, Pauli, Klein-Gordon, or Dirac), while the electromagnetic field is considered classically based on Maxwell equations. Semi-classical theories allow describing many basic quantum effects, such as the Compton effect [1−4,18], spontaneous emission [6−11,19,24] and spontaneous transitions [15−17,19], light–atom interactions [7,20], the photoelectric effect [5,21], thermal radiation [22,25], induced emission [6,7,20,22], the Lamb shift [6,7,9], and the Lamb–Retherford experiments [23], g-2 factor [12−14], among others. Thus, in fact, the myth is dispelled that the so-called quantum processes cannot be described within the framework of classical physics, and this necessarily requires quantization of both the matter itself (atoms and electrons) and the electromagnetic fields interacting with it.

At the same time, most semi-classical theories are characterized by inconsistency: having rejected the quantization of the electromagnetic field, they retain (sometimes implicitly) quantum concepts in relation to non-relativistic matter (electrons).

In [18−23], it was shown that it is possible to construct a completely classical theory that consistently and unambiguously describes and explains all basic quantum effects within the framework of classical field theory without any quantization or other additional hypotheses. This theory is based on the original Schrödinger's idea [26,27] that the wave functions ψ, described by wave equations, allow constructing real parameters that can be interpreted as the density of electric charge and the density of electric current, continuously distributed in space. Taking this into account, at least from a formal mathematical point of view, one can speak of some real electrically charged material (for example, electron) field ψ, continuously distributed in space, which is described by a wave equation similar to how the classical electromagnetic field is described by Maxwell equations [18,19]. Further, such a field ψ will be conventionally called an electron field or an electron wave. According to classical electrodynamics, charges and currents continuously distributed in space create an electromagnetic field, which, in turn, must act on them, changing the field ψ, and therefore, its charge density and current density. Thus, the potentials of the electromagnetic field included in the wave equations of quantum mechanics must be a superposition of the potentials of the external electromagnetic field (created by external sources) and the potentials of the own electromagnetic field created by the electrically charged material field ψ.

However, linear wave equations (Schrödinger, Pauli, Klein–Gordon, Dirac), which are the basis of modern quantum mechanics, contain the external electromagnetic field but do not include their own electromagnetic field created by the electrically charged electron wave of the atom. At the same time, if we consider the interaction of an atom with another atom, then the wave equation for the atom under consideration will contain the electromagnetic field created by the electron wave of the other atom. A paradox arises: the electron wave of the atom feels the electrostatic field created by the electron wave of another atom but does not feel its own electrostatic field [18−23].

In the initial formulation of the theory [18−23], the own electrostatic field of the electron wave in the atom was simply not taken into account, but the external electromagnetic field was considered, i.e., the field created by external sources, including the electron waves of other atoms and the own nonstationary (radiative) electromagnetic field created by the electron wave under consideration. The latter, as shown in [19−23], is responsible for the so-called spontaneous transitions.

As shown in [18−23], this point of view allows naturally describing many quantum effects

without quantization and without involving such hypothetical entities as the quantum-electrodynamic vacuum, its fluctuations, Zitterbewegung, etc.

To explain why the electron wave in an atom does not feel its own electrostatic field, the assumption was made in [28] that there is an unknown short-range field that compensates for the electron wave's own electrostatic field but does not compensate for its nonstationary (radiative) field, as well as the electromagnetic fields created by electron waves of other atoms.

The Maxwell-Pauli theory was constructed in [28], based on a system of equations that has the necessary properties:

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[\frac{1}{2m_e}\left(\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}_\Sigma\right)^2-e\varphi_\Sigma-e\boldsymbol{\sigma}\mathbf{G}_\Sigma+\frac{e\hbar}{2m_ec}\boldsymbol{\sigma}\mathbf{H}_\Sigma\right]\Psi \tag{1}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{G}_\Sigma}{\partial t^2}-\Delta\mathbf{G}_\Sigma+\varkappa^2\mathbf{G}_\Sigma=4\pi e(\Psi^*\boldsymbol{\sigma}\Psi) \tag{2}$$

$$\mathbf{H}_\Sigma=\mathbf{H}+\mathbf{H}_e,\mathbf{E}_\Sigma=\mathbf{E}+\mathbf{E}_e,\mathbf{G}_\Sigma=\mathbf{G}+\mathbf{G}_e \tag{3}$$

$$\varphi_\Sigma=\varphi+\varphi_e,\ \mathbf{A}_\Sigma=\mathbf{A}+\mathbf{A}_e \tag{4}$$

$$\mathrm{rot}\mathbf{H}_\Sigma=\frac{1}{c}\frac{\partial\mathbf{E}_\Sigma}{\partial t}+\frac{4\pi}{c}\mathbf{j} \tag{5}$$

$$\mathrm{div}\mathbf{E}_\Sigma=4\pi\rho \tag{6}$$

where

$$\mathbf{H}_\Sigma=\mathrm{rot}\mathbf{A}_\Sigma,\ \mathbf{E}_\Sigma=-\frac{1}{c}\frac{\partial\mathbf{A}_\Sigma}{\partial t}-\nabla\varphi_\Sigma \tag{7}$$

$$\rho=-e\Psi^*\Psi \tag{8}$$

$$\mathbf{j}=\frac{e\hbar}{2m_ei}\left[(\nabla\Psi^*)\Psi-\Psi^*\nabla\Psi\right]-\frac{e^2}{m_ec}\mathbf{A}_\Sigma\Psi^*\Psi-\frac{e\hbar}{2m_e}\mathrm{rot}(\Psi^*\boldsymbol{\sigma}\Psi) \tag{9}$$

the index Σ refers to the total fields; the index e refers to the own fields created by the electron wave, which is described by the spinor $\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$; parameters without an index refer to external fields created by external (with respect to the field Ψ) charges, currents, and spins; m_e is the electron mass; $\varkappa$ is a constant satisfying condition [28]

$$\varkappa a_B\ll 1 \tag{10}$$

where $a_B=\frac{\hbar^2}{m_ee^2}$ is the Bohr radius. The constant $\varkappa$, satisfying condition (10), makes the $\mathbf{G}$ field short-range, exponentially decaying at distances of the order of a_B from the source. This explains why the $\mathbf{G}$ field has not been experimentally detected to date, and also why the electron field of an atom does not "feel" its own electrostatic field, but "feels" electrostatic fields created by other atoms and ions.

The system of equations (1)–(9) is closed and self-consistent. It differs from a simple formal unification of the Maxwell and Pauli equations in that the Pauli equation (1) includes an additional

term, $-e\boldsymbol{\sigma}\mathbf{G}_{\boldsymbol{\Sigma}}$, which plays an important role. In this paper, we will show that this fundamentally changes the solutions of the unified system of Maxwell-Pauli equations and allows correct description of quantum effects without any additional hypotheses and postulates.

The system of equations (1)–(9) has gauge invariance:

$$\mathbf{A}_{\boldsymbol{\Sigma}} \to \mathbf{A}_{\boldsymbol{\Sigma}} + \nabla f,\ \varphi_{\boldsymbol{\Sigma}} \to \varphi_{\boldsymbol{\Sigma}} - \frac{1}{c}\frac{\partial f}{\partial t},\ \Psi \to \Psi \exp\left(-\frac{ie}{\hbar c} f\right) \tag{11}$$

where f is an arbitrary function; in this case, the electric and magnetic field strengths, the field $\mathbf{G}_{\boldsymbol{\Sigma}}$, the electric charge density, the current density, and other physical characteristics of the electron field do not change.

Equation (1), which describes the charged field Ψ, follows the law of conservation of the electric charge of the electron wave

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0 \tag{12}$$

which is known to be consistent with Maxwell's equations.

As shown in [28], equations (1)–(9) yield the theory [18−23], and therefore, the Maxwell-Pauli theory allows naturally describing many quantum effects without quantization and without involving such hypothetical entities as the quantum-electrodynamic vacuum, its fluctuations, Zitterbewegung, etc.

Obviously, the Maxwell-Pauli theory (1)–(9) is not relativistically invariant. A relativistically invariant theory (the Maxwell–Dirac theory) was constructed in [29]. As shown in [29], the Maxwell-Pauli theory (1)-(9) is a non-relativistic limit of the Maxwell–Dirac theory.

In [30], solutions of the system of equations (1)–(9) were investigated for $\varkappa = 0$, i.e., for a long-range field $\mathbf{G}$.

In this paper, we consider the general case $\varkappa \neq 0$, when the compensating field $\mathbf{G}$ is short-range and decays at distances of the order of $\varkappa^{-1} a_B$.

First of all, we will be interested in the spontaneous emission of an atom. We will show that it is a natural consequence of theory (1)–(9) without any additional hypotheses and postulates. This analysis is based on the well-known results of classical electrodynamics, in particular, the theory of dipole radiation of charged matter. As is known [31], the intensity of electric dipole radiation

$$I_{\mathbf{EH}} = \frac{2}{3c^3}\ddot{\mathbf{d}}^2 \tag{13}$$

where

$$\mathbf{d} = \int \mathbf{r}\rho\, dV \tag{14}$$

is the dipole moment of the electron field.

According to [28], the nonstationary field $\mathbf{G}$ can also transfer energy, i.e., it has an energy flux, the density of which is

$$\mathbf{J}_{\mathbf{G}} = -\frac{1}{4\pi}\frac{\partial G_{\Sigma k}}{\partial t}\nabla G_{\Sigma k} \tag{15}$$

Accordingly, the radiation intensity of the $\mathbf{G}$ field

$$I_{\mathbf{G}} = \oint \mathbf{J}_{\mathbf{G}} d\mathbf{S} \tag{16}$$

where the integral is taken over a spherical surface of infinite radius with its center at the nucleus of the atom.

Then the total intensity of spontaneous emission (electromagnetic and **G** field) of the atom (the rate at which the atom loses energy in the process of spontaneous emission) is equal to

$$I = I_{\mathbf{EH}} + I_{\mathbf{G}} \tag{17}$$

2. Nonlinear Pauli equation

In [30], the nonlinear Pauli equation was derived for the case $\varkappa = 0$. In this work, we consider the general case $\varkappa \neq 0$.

The scalar and vector potentials of the own electromagnetic field of an electron wave are determined by the known solutions of Maxwell equations [31]

$$\varphi_e = \int \frac{\rho_{t-R/c}}{R} dV' \tag{18}$$

$$\mathbf{A}_e = \frac{1}{c} \int \frac{\mathbf{j}_{t-R/c}}{R} dV' \tag{19}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

In this case, the Lorentz calibration takes place [31]

$$\frac{1}{c} \frac{\partial \varphi_e}{\partial t} + \mathrm{div} \mathbf{A}_e = 0 \tag{20}$$

Let us expand the integrands in (18) and (19) in a series in powers of the parameter R/c [31]:

$$\varphi_e = \varphi_0 - \frac{1}{c} \frac{\partial}{\partial t} \int \rho dV' + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int R \rho dV' - \frac{1}{6c^3} \frac{\partial^3}{\partial t^3} \int R^2 \rho dV' + \cdots \tag{21}$$

$$\mathbf{A}_e = \mathbf{A}_0 - \frac{1}{c^2} \frac{\partial}{\partial t} \int \mathbf{j} dV' + \frac{1}{2c^3} \frac{\partial^2}{\partial t^2} \int R \mathbf{j} dV' - \cdots \tag{22}$$

where ρ and $\mathbf{j}$ are taken at time t;

$$\mathbf{A}_0 = \frac{1}{c} \int \frac{\mathbf{j}}{R} dV' \tag{23}$$

$$\varphi_0 = \int \frac{\rho}{R} dV' \tag{24}$$

Let us take into account that

$$q = \int \rho dV \tag{25}$$

is the total electric charge of the electron field. The change in the total electric charge of a localized (for example, in an atom) electron field can occur as a result of the ionization of the atom, i.e., either as a result of the emission of an electron wave by the atom or as a result of the capture of an electron wave by the atom from the outside. These processes are not considered in this paper, i.e., it is

assumed that the total electric charge of the electron field does not change:

$$\frac{\partial q}{\partial t} = 0 \tag{26}$$

In this case, it can only be redistributed between excited modes and polarizations of the atom [19−23,28].

From the continuity equation (12), it follows that for a localized electron field

$$\int \mathbf{j} dV = \dot{\mathbf{d}} \tag{27}$$

Taking into account (26) and (27), one writes relations (21) and (22) in the form

$$\varphi_e = \varphi_0 + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' - \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\rho dV' + \cdots \tag{28}$$

$$\mathbf{A}_e = \mathbf{A}_0 - \frac{1}{c^2}\ddot{\mathbf{d}} + \frac{1}{2c^3}\frac{\partial^2}{\partial t^2}\int R\mathbf{j} dV' - \cdots \tag{29}$$

Let us calculate

$$\frac{\partial^3}{\partial t^3}\int R^2\rho dV' = \frac{\partial^3}{\partial t^3}\int (\mathbf{r}-\mathbf{r}')^2\rho dV' = \frac{\partial^3}{\partial t^3}\int (\mathbf{r}^2 - 2\mathbf{r}\mathbf{r}' + \mathbf{r}'^2)\rho dV' = \mathbf{r}^2\frac{\partial^3}{\partial t^3}\int \rho dV' - 2\mathbf{r}\frac{\partial^3}{\partial t^3}\int \mathbf{r}'\rho dV' + \frac{\partial^3}{\partial t^3}\int \mathbf{r}'^2\rho dV'$$

Taking into account (14), (25), and (26), one obtains

$$\frac{\partial^3}{\partial t^3}\int R^2\rho dV' = -2\mathbf{r}\ddot{\mathbf{d}} + \frac{\partial^3}{\partial t^3}\int \mathbf{r}'^2\rho dV' \tag{30}$$

Let us perform the gauge transformation (11) with the function

$$f = \frac{1}{c^2}\mathbf{r}\ddot{\mathbf{d}} - \frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int \mathbf{r}'^2\rho dV' \tag{31}$$

where the last term does not depend on $\mathbf{r}$.

As a result, taking into account (30), the scalar and vector potentials of the electromagnetic field (28) and (29) take the form

$$\varphi_e = \varphi_0 - \frac{2}{3c^3}\mathbf{r}\ddot{\mathbf{d}} + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' + \cdots \tag{32}$$

$$\mathbf{A}_e = \mathbf{A}_0 + \frac{1}{2c^3}\frac{\partial^2}{\partial t^2}\int R\mathbf{j} dV' - \cdots \tag{33}$$

Let us estimate the different terms in (32) and (33) for the electron wave inside the atom, taking into account that the characteristic spatial scale of change of the functions ρ and $\mathbf{j}$ is the Bohr radius $a_B = \frac{\hbar^2}{m_e e^2}$, and the characteristic time of their change is $\omega^{-1} = \frac{a_B}{\alpha c}$, where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant. Taking into account (9), one obtains the estimate $\mathbf{j} \sim \frac{\hbar}{m_e a_B}\rho = \alpha c\rho$. Then, taking

into account that $\frac{a_B\omega}{c}=\alpha$, one obtains $\varphi_0\sim\frac{e}{a_B}$, $\frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV'\sim\frac{e}{a_B}\left(\frac{a_B\omega}{c}\right)^2=\alpha^2\frac{e}{a_B}$, $\frac{2}{3c^3}\mathbf{r}\ddot{\mathbf{d}}\sim\frac{e}{a_B}\left(\frac{a_B\omega}{c}\right)^3=\alpha^3\frac{e}{a_B}$, $\mathbf{A}_0\sim\alpha\frac{e}{a_B}$, $\frac{1}{2c^3}\frac{\partial^2}{\partial t^2}\int R\mathbf{j}dV'\sim\frac{e}{a_B}\alpha\left(\frac{a_B\omega}{c}\right)^2=\alpha^3\frac{e}{a_B}$, etc. In addition, in equation (1), the term $\frac{\hbar e}{m_e c}\mathbf{A}_e\nabla\sim\frac{\hbar e}{m_e c a_B}\mathbf{A}_0\sim\alpha^2\frac{e^2}{a_B}$. The remaining nonstationary components of the field (33) lead in equation (1) to terms of order α^4 and higher compared to the main term $\varphi_0\sim\frac{e^2}{a_B}$ and can be discarded.

Restricting ourselves in equation (1) to terms of order $\frac{e}{a_B}\alpha^3$, one writes (32) and (33) in the form

$$\varphi_e=\varphi_0-\frac{2}{3c^3}\mathbf{r}\ddot{\mathbf{d}}+\frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' \tag{34}$$

$$\mathbf{A}_e=\mathbf{A}_0 \tag{35}$$

Neglecting diamagnetic effects, we leave in equation (1) only the terms linear in $\mathbf{A}_\Sigma$. Then, taking into account (34) and (35), one writes equation (1) in the form

$$i\hbar\frac{\partial\Psi}{\partial t}=\Big[-\frac{\hbar^2}{2m_e}\Delta-e\varphi+\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}}-e(\varphi_0+\boldsymbol{\sigma}\mathbf{G}_e)-e\boldsymbol{\sigma}\mathbf{G}+\frac{e\hbar}{i2m_e c}(\nabla\mathbf{A})+\frac{e\hbar}{im_e c}(\mathbf{A}\nabla)+\frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\mathbf{H}+$$
$$\frac{e\hbar}{i2m_e c}(\nabla\mathbf{A}_0)+\frac{e\hbar}{im_e c}(\mathbf{A}_0\nabla)-\frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV'+\frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\mathbf{H}_0\Big]\Psi \tag{36}$$

where

$$\mathbf{H}_0=\mathrm{rot}\mathbf{A}_0 \tag{37}$$

We will assume that the external magnetic field changes insignificantly over distances of the order of a_B. Then we can write

$$\mathbf{A}=\frac{1}{2}\mathbf{H}\times\mathbf{r} \tag{38}$$

In this case, taking into account (20), one writes equation (36) in the form

$$i\hbar\frac{\partial\Psi}{\partial t}=\Big[-\frac{\hbar^2}{2m_e}\Delta-e\varphi+\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}}-e(\varphi_0+\boldsymbol{\sigma}\mathbf{G}_e)-e\boldsymbol{\sigma}\mathbf{G}+\frac{e\hbar}{i2m_e c}\mathbf{H}(\mathbf{r}\times\nabla)+\frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\mathbf{H}-$$
$$\frac{e\hbar}{i2m_e c^2}\frac{\partial\varphi_0}{\partial t}+\frac{e\hbar}{im_e c}(\mathbf{A}_0\nabla)-\frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV'+\frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\mathbf{H}_0\Big]\Psi \tag{39}$$

Taking into account the estimate

$$\frac{e^2}{m_e c}\mathbf{A}_e\Psi^*\Psi\sim\frac{e^2}{m_e c}\frac{\mathbf{1}}{\boldsymbol{c}}a_B^2\mathbf{j}a_B^{-3}=\frac{e^2}{m_e c^2 a_B}\mathbf{j}=\frac{e^4}{c^2\hbar^2}\mathbf{j}=\alpha^2\mathbf{j} \tag{40}$$

with the accuracy considered, the current density of the electron wave can be written in the form

$$\mathbf{j}=\mathbf{j}_c-\frac{e\hbar}{2m_e}\mathrm{rot}(\Psi^*\boldsymbol{\sigma}\Psi) \tag{41}$$

where

$$\mathbf{j}_c = \frac{e\hbar}{2m_e i}\,[(\nabla\Psi^*)\Psi - \Psi^*\nabla\Psi] \tag{42}$$

is the convective component of the current density [28]. Note that in the current density (41), not only the term containing the intrinsic field $\mathbf{A}_0$ is discarded but also the term containing the external field $\mathbf{A}$ (38). This is due to the fact that its contribution (through $\mathbf{A}_0$ and $\mathbf{H}_0$) to equation (39) is of the order of α^2 compared to the corresponding terms of equation (39) already containing $\mathbf{H}$, which is beyond the scope of the approximation under consideration.

Taking into account (41), equation (23) takes the form

$$\mathbf{A}_0 = \mathbf{A}_c - \frac{e\hbar}{2m_e c}\int \frac{\mathrm{rot}(\Psi^*\boldsymbol{\sigma}\Psi)}{R}\,dV' \tag{43}$$

where

$$\mathbf{A}_c = \frac{1}{c}\int \frac{\mathbf{j}_c}{R}\,dV \tag{44}$$

For a localized electron field one writes (43) in the form

$$\mathbf{A}_0 = \mathbf{A}_c - \frac{e\hbar}{2m_e c}\nabla \times \int \frac{(\Psi^*\boldsymbol{\sigma}\Psi)}{R}\,dV' \tag{45}$$

Taking into account (37) and (44), one obtains

$$\mathbf{H}_0 = \mathrm{rot}\mathbf{A}_c - \frac{e\hbar}{2m_e c}\nabla\left(\mathrm{div}\int \frac{(\Psi^*\boldsymbol{\sigma}\Psi)}{R}\,dV'\right) + \frac{e\hbar}{2m_e c}\Delta\int \frac{(\Psi^*\boldsymbol{\sigma}\Psi)}{R}\,dV' \tag{46}$$

Taking into account, that $\Delta\frac{1}{R} = -4\pi\delta(\mathbf{r}-\mathbf{r}')$, where $\delta(\mathbf{r}-\mathbf{r}')$ is the delta function, one obtains

$$\mathbf{H}_0 = \mathrm{rot}\mathbf{A}_c - \frac{e\hbar}{2m_e c}\nabla\left(\mathrm{div}\int \frac{(\Psi^*\boldsymbol{\sigma}\Psi)}{R}\,dV'\right) - 4\pi\frac{e\hbar}{2m_e c}(\Psi^*\boldsymbol{\sigma}\Psi) \tag{47}$$

Equation (39), taking into account (14), (24), (44), (45), (47), and the solution of equation (2), is the nonlinear Pauli equation.

It differs from the nonlinear Pauli equation obtained in [30] by the factor 2/3 in the term $\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}}$. As will be shown below, this is a consequence of the condition $\varkappa \neq 0$ and fundamentally changes the solutions of equation (39).

3. Particular solutions of equation (2)

Before considering the solution of equation (39), let us consider nonstationary solutions of the inhomogeneous Klein–Gordon equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u + \varkappa^2 u = 4\pi(f_0 + f_1\exp(\mp i\omega t)) \tag{48}$$

where f_0 and f_1 are the functions of only spatial coordinates; $\omega > 0$ is the constant.

The solution of equation (48) can be represented as

$$u = u_0 + u_1 \exp(\mp i\omega t) \tag{49}$$

where u_0 and u_1 are the functions of only spatial coordinates that satisfy the equations

$$-\Delta u_0 + \varkappa^2 u_0 = 4\pi f_0 \tag{50}$$

$$-\Delta u_1 + \left(\varkappa^2 - \frac{\omega^2}{c^2}\right) u_1 = 4\pi f_1 \tag{51}$$

Equation (50) has a solution

$$u_0 = \int f_0 \frac{\exp(-\varkappa R)}{R} dV' \tag{52}$$

The solution of equation (51) depends on the sign of the constant $\left(\varkappa^2 - \frac{\omega^2}{c^2}\right)$.

At $\omega < \varkappa c$ one obtains

$$u_1 = \int f_1 \frac{\exp\left(-\sqrt{\varkappa^2 - \frac{\omega^2}{c^2}} R\right)}{R} dV' \tag{53}$$

Here, we consider only solutions (52) and (53), decreasing at infinity.

Solution (53) corresponds to oscillatory non-wave fields (all points of the field oscillate in the same phase).

At $\omega > \varkappa c$, one obtains

$$u_1 = \int f_1 \frac{\exp\left(\pm i\sqrt{\frac{\omega^2}{c^2} - \varkappa^2} R\right)}{R} dV' \tag{54}$$

The sign in solution (54) is opposite to the sign in front of ω in (49), since we consider only solutions corresponding to retarded wave fields. This means that in the limit $\varkappa \to 0$, equation (48) transforms into the d'Alembert wave equation, the retarded solution of which contains the phase $\omega(t - R/c)$. It is precisely into this solution that solution (54) must pass in the limit $\varkappa \to 0$. These fields propagate with the group velocity $v = \frac{\partial \omega}{\partial \sqrt{\frac{\omega^2}{c^2} - \varkappa^2}} = c\sqrt{1 - \frac{\varkappa^2 c^2}{\omega^2}}$.

If the field u is real and its source has a discrete frequency spectrum $\omega_n \neq 0$, then equation (48) takes the form

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u + \varkappa^2 u = 4\pi\left(f_0 + \sum_{\omega_n} f_n \exp(-i\omega_n t) + \sum_{\omega_n} f_n^* \exp(i\omega_n t)\right) \tag{55}$$

where f_0 and f_n – and f_1 are the functions of only spatial coordinates; f_0 is the real function.

Taking into account (52)–(54), one writes the solution of equation (55) in the form

$$u = \int f_0 \frac{\exp(-\varkappa R)}{R} dV' + \sum_{\omega_n<\varkappa c} \exp(-i\omega_n t) \int f_n \frac{\exp\left(-\sqrt{\varkappa^2-\frac{\omega_n^2}{c^2}}R\right)}{R} dV' +$$

$$\sum_{\omega_n<\varkappa c} \exp(i\omega_n t) \int f_n^* \frac{\exp\left(-\sqrt{\varkappa^2-\frac{\omega_n^2}{c^2}}R\right)}{R} dV' + \sum_{\omega_n>\varkappa c} \exp(-i\omega_n t) \int f_n \frac{\exp\left(i\sqrt{\frac{\omega_n^2}{c^2}-\varkappa^2}R\right)}{R} dV' +$$

$$\sum_{\omega_n>\varkappa c} \exp(i\omega_n t) \int f_n^* \frac{\exp\left(-i\sqrt{\frac{\omega_n^2}{c^2}-\varkappa^2}R\right)}{R} dV' \tag{56}$$

4. Nonlinear Schrödinger equation for the hydrogen atom

Consider a hydrogen atom in empty space ($\varphi = \frac{e}{r}$, $\mathbf{A} = 0$ and $\mathbf{G} = 0$). In this case, equation (39) takes the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m_e}\Delta - \frac{e^2}{r} + \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e(\varphi_0 + \boldsymbol{\sigma}\mathbf{G}_e) - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' - \frac{e\hbar}{i2m_e c^2}\frac{\partial \varphi_0}{\partial t} + \frac{e\hbar}{im_e c}(\mathbf{A}_0\nabla) + \frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\mathbf{H}_0\right]\Psi \tag{57}$$

According to the estimates made above, the third, fifth, sixth, seventh, and eighth terms on the right-hand side of equation (57) are small corrections of the order of α^2 and higher compared to the first two terms on the right-hand side of equation (57).

We neglect the last three terms in equation (57), associated with the intrinsic magnetic field. Then equation (57) takes the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m_e}\Delta - \frac{e^2}{r} + \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e(\varphi_0 + \boldsymbol{\sigma}\mathbf{G}_e) - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV'\right]\Psi \tag{58}$$

The vector $\Psi^*\boldsymbol{\sigma}\Psi$ can always be represented in the form [28]

$$\Psi^*\boldsymbol{\sigma}\Psi = \boldsymbol{\nu}\Psi^*\Psi \tag{59}$$

where $\boldsymbol{\nu}$ is the unit vector ($\boldsymbol{\nu}^2 = 1$), which in general can depend on time and spatial coordinates.

Equation (58) allows a solution of the form [28]

$$\Psi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\psi \tag{60}$$

where a_1 and a_2 are the constants satisfying the condition

$$a_1^2 + a_2^2 = 1 \tag{61}$$

while the function $\psi(t, \mathbf{r})$ satisfies the normalization condition

$$\int |\psi|^2 dV = 1 \tag{62}$$

which has a simple deterministic physical meaning [18−23], not related to probability: for a neutral hydrogen atom, the total electric charge of the electron wave is $-e$.

In this case, equation (59) takes the form

$$\Psi^*\boldsymbol{\sigma}\Psi = \mathbf{\nu}|\psi|^2 \tag{63}$$

where

$$\mathbf{\nu} = (a_1^*a_2 + a_1a_2^*, i(a_1a_2^* - a_1^*a_2), |a_1|^2 - |a_2|^2) \tag{64}$$

is a constant unit vector, and the solution of equation (2) has the form

$$\mathbf{G}_e = \mathbf{\nu}u \tag{65}$$

where the function $u(t, \mathbf{r})$ satisfies the inhomogeneous Klein–Gordon equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u + \varkappa^2 u = 4\pi e|\psi|^2 \tag{66}$$

The energy flux density of the **G** field (15) for case (65) takes the form

$$\mathbf{J}_\mathbf{G} = -\frac{1}{4\pi}\frac{\partial u}{\partial t}\nabla u \tag{67}$$

Taking into account (60), (64), and (65), it is easy to verify directly that

$$\boldsymbol{\sigma}\mathbf{G}_e\Psi = \binom{a_1}{a_2} u\psi \tag{68}$$

Note that equation (57) allows a solution of the form (60) even in a more general case, if only the last term can be neglected.

In the approximation under consideration, equation (58) takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m_e}\Delta - \frac{e^2}{r} + \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV'\right]\psi \tag{69}$$

for any values of constants a_1 and a_2.

Let us consider a hydrogen atom in a mixed state, in which several eigenmodes of the electron wave are excited simultaneously [19−23].

In this case, the function $\psi(t, \mathbf{r})$ can be represented as

$$\psi(\mathbf{r}, t) = \sum_n c_n\psi_n(\mathbf{r})\exp(-i\omega_n t) \tag{70}$$

where the subscript n denotes a set of quantum numbers (n, l, m); c_n are constants characterizing the excitation amplitude of the eigenmode n; ω_n are the eigenfrequencies of the nonlinear equation (69), which will be defined below; $\psi_n(\mathbf{r})$ and ω_n^0 are the eigenfunctions and eigenvalues (eigenfrequencies) of the stationary linear Schrödinger equation

$$\hbar\omega_n^0\psi_n = -\left(\frac{\hbar^2}{2m_e}\Delta + \frac{e^2}{r}\right)\psi_n \tag{71}$$

The functions $\psi_n(\mathbf{r})$ form an orthonormal basis:

$$\int \psi_n^*\psi_k dV = \delta_{nk} \tag{72}$$

Expansion (70) has a simple physical meaning: the hydrogen atom has a discrete set of eigenmodes, which have eigenfrequencies ω_n, and are described in space by eigenfunctions $\psi_n(\mathbf{r})$. This means that, at least formally, the hydrogen atom can be considered as a classical volume

resonator, which holds an electron wave and has a discrete set of eigenmodes and their corresponding eigenfrequencies [19−23].

Then, taking into account (62) and (72), one obtains

$$\sum_n |c_n|^2 = 1 \tag{73}$$

The parameter c_n describes the distribution of the electric charge of the electron wave over the eigenmodes of the hydrogen atom. In particular, the value $-e|c_n|^2$ is equal to the electric charge of the electron wave contained in mode n [19−23].

Using expansion (70), one obtains

$$|\psi|^2 = \sum_{k,n} c_k^* c_n \psi_k^*(\mathbf{r})\psi_n(\mathbf{r}) \exp(-i\omega_{nk}t) \tag{74}$$

where

$$\omega_{nk} = \omega_n - \omega_k \tag{75}$$

Function (74) can be represented as

$$|\psi|^2 = \sum_n |c_n|^2 |\psi_n(\mathbf{r})|^2 + \sum_{\substack{\omega_n=\omega_k \\ n\neq k}} c_k^* c_n \psi_k^*(\mathbf{r})\psi_n(\mathbf{r}) +$$

$$\sum_{\omega_n>\omega_k} c_k^* c_n \psi_k^*(\mathbf{r})\psi_n(\mathbf{r}) \exp(-i\omega_{nk}t) + \sum_{\omega_n>\omega_k} c_n^* c_k \psi_n^*(\mathbf{r})\psi_k(\mathbf{r}) \exp(i\omega_{nk}t) \tag{76}$$

The first two sums in the approximation under consideration are independent of time. The second sum corresponds to degenerate modes.

Taking into account (76), one writes (24) and (14) in the form

$$\varphi_0 = -e \int \left(\sum_n |c_n|^2 |\psi_n|^2 + \sum_{\substack{\omega_n=\omega_k \\ n\neq k}} c_k^* c_n \psi_k^* \psi_n + \sum_{\omega_n>\omega_k} c_k^* c_n \psi_k^* \psi_n \exp(-i\omega_{nk}t) + \right.$$

$$\left. \sum_{\omega_n>\omega_k} c_n^* c_k \psi_n^* \psi_k \exp(i\omega_{nk}t) \right) \frac{1}{R} dV' \tag{77}$$

$$\mathbf{d}(t) = \sum_k \sum_n c_k^* c_n \mathbf{d}_{nk} \exp(-i\omega_{nk}t) \tag{78}$$

where

$$\mathbf{d}_{nk} = \mathbf{d}_{kn}^* = -e \int \mathbf{r} \psi_k^*(\mathbf{r})\psi_n(\mathbf{r}) dV \tag{79}$$

Taking into account (78), one obtains

$$\ddot{\mathbf{d}}(t) = -\sum_k \sum_n \omega_{nk}^2 c_k^* c_n \mathbf{d}_{nk} \exp(-i\omega_{nk}t) \tag{80}$$

$$\dddot{\mathbf{d}}(t) = i \sum_k \sum_n \omega_{nk}^3 c_k^* c_n \mathbf{d}_{nk} \exp(-i\omega_{nk}t) \tag{81}$$

Using (76) and solution (56) of equation (55), one writes the solution of equation (66) in the form

$$u = e \int \left[\sum_n |c_n|^2 |\psi_n|^2 + \sum_{\substack{\omega_n=\omega_k \\ n\neq k}} c_k^* c_n \psi_k^* \psi_n \right] \frac{\exp(-\varkappa R)}{R} dV' +$$

$$e \sum_{\substack{\omega_n>\omega_k \\ \omega_{nk}<\varkappa c}} c_k^* c_n \exp(-i\omega_{nk}t) \int \psi_k^* \psi_n \frac{\exp\left(-\sqrt{\varkappa^2 - \frac{\omega_{nk}^2}{c^2}} R \right)}{R} dV' +$$

$$e\sum_{\substack{\omega_n>\omega_k\\ \omega_{nl}<\varkappa c}} c_n^* c_k \exp(i\omega_{nk}t)\int \psi_n^*\psi_k \frac{\exp\left(-\sqrt{\varkappa^2-\frac{\omega_{nk}^2}{c^2}}R\right)}{R}dV' +$$

$$e\sum_{\substack{\omega_n>\omega_k\\ \omega_{nk}>\varkappa c}} c_k^* c_n \exp(-i\omega_{nk}t)\int \psi_k^*\psi_n \frac{\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV' +$$

$$e\sum_{\substack{\omega_n>\omega_k\\ \omega_{nk}>\varkappa c}} c_n^* c_k \exp(i\omega_{nk}t)\int \psi_n^*\psi_k \frac{\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV' \tag{82}$$

Taking into account (76), (77), and (82), one writes

$$\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' = \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} + e^2\int\left(\sum_n |c_n|^2|\psi_n|^2 + \sum_{\substack{\omega_n=\omega_k\\ n\neq k}} c_k^* c_n \psi_k^*\psi_n\right)\frac{1-\exp(-\varkappa R)}{R}dV' +$$

$$e^2\sum_{\substack{\omega_n>\omega_k\\ \omega_{nk}<\varkappa c}} c_k^* c_n \exp(-i\omega_{nk}t)\int \psi_k^*\psi_n \frac{1-\exp\left(-\sqrt{\varkappa^2-\frac{\omega_{nk}^2}{c^2}}R\right)}{R}dV' +$$

$$e^2\sum_{\substack{\omega_n>\omega_k\\ \omega_{nk}>\varkappa c}} c_k^* c_n \exp(-i\omega_{nk}t)\int \psi_k^*\psi_n \frac{1-\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV' +$$

$$e^2\sum_{\substack{\omega_n>\omega_k\\ \omega_{nl}<\varkappa c}} c_n^* c_k \exp(i\omega_{nk}t)\int \psi_n^*\psi_k \frac{1-\exp\left(-\sqrt{\varkappa^2-\frac{\omega_{nk}^2}{c^2}}R\right)}{R}dV' +$$

$$e^2\sum_{\substack{\omega_n>\omega_k\\ \omega_{nk}>\varkappa c}} c_n^* c_k \exp(i\omega_{nk}t)\int \psi_n^*\psi_k \frac{1-\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV' -$$

$$\frac{e^2}{2c^2}\sum_{\omega_n>\omega_k}\omega_{nk}^2 c_k^* c_n \exp(-i\omega_{nk}t)\int R\psi_k^*\psi_n dV' - \frac{e^2}{2c^2}\sum_{\omega_n>\omega_k}\omega_{nk}^2 c_n^* c_k \exp(i\omega_{nk}t)\int R\psi_n^*\psi_k dV' \tag{83}$$

Equation (69), together with relations (81) and (83), represents a nonlinear Schrödinger equation with a short-range compensating field.

5. Stationary states of the atom

The stationary state of an atom is a state in which it does not lose energy due to emission, i.e.,

$$I_{\mathbf{EH}} = I_{\mathbf{G}} = 0 \tag{84}$$

From a mathematical point of view, this means that all physical parameters of the electron wave in the atom (charge density, current density, density of proper angular momentum, $\mathbf{G}$ field, etc.) are independent of time, and accordingly, the terms on the right-hand side of the equation (69) containing time derivatives are equal to zero. The stationary state of the atom can be realized only in three cases: (i) when the atom is in a pure state, i.e., it is in a state with one excited eigenmode n ($c_n = 1, c_k = 0$ for all $k \neq n$); (ii) when the atom is in a mixed state with several simultaneously excited eigenmodes, and all excited modes have the same proper frequency ω_n, i.e., are degenerate; (iii) when the atom is in a mixed state with several simultaneously excited non-degenerate modes, and for any pair of excited modes n and k, the condition [19−23]

$$\mathbf{d}_{nk} = 0$$

is satisfied (in quantum mechanics, following a tradition dating back to Bohr naive theory, such pairs are called “forbidden transitions”).

If the atom is in a pure state n, then (83) takes the form

$$-e\varphi_0 - eu = e^2 \int |\psi_n|^2 \frac{1-\exp(-\varkappa R)}{R} dV' \tag{85}$$

Under condition (10), expanding the exponential under the integral in (85) in a series in powers of $\varkappa R$, taking into account (72), one approximately obtains

$$-e\varphi_0 - eu = \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int R|\psi_n|^2 dV' + \frac{1}{6}\varkappa^3 e^2 \int R^2 |\psi_n|^2 dV' \tag{86}$$

Expanding the last integral in (86), and taking into account (79), one obtains

$$-e\varphi_0 - eu = \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int R|\psi_n|^2 dV' + \frac{1}{3}\varkappa^3 e\mathbf{r}\mathbf{d}_{nn} + \frac{1}{6}\varkappa^3 e^2 \mathbf{r}^2 + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}'^2 |\psi_n|^2 dV' \tag{87}$$

Substituting (87) into the Schrödinger equation (69) for the non-degenerate stationary state $\psi = \psi_n \exp(-i\omega_n t)$ of the hydrogen atom, and using perturbation theory [32], taking into account equation (71), one obtains

$$\hbar(\omega_n - \omega_n^0) = \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int\int |\psi_n(\mathbf{r})|^2 |\mathbf{r} - \mathbf{r}'| |\psi_n(\mathbf{r}')|^2 dV' dV + \frac{1}{3}\varkappa^3 (e^2 \int \mathbf{r}^2 |\psi_n|^2 dV - \mathbf{d}_{nn}^2) \tag{88}$$

As follows from (88), for $\varkappa \neq 0$, the eigenfrequencies of various atomic modes differ from the eigenfrequencies of the linear Schrödinger equation (71); however, under condition (10), this difference is small: $\omega_n - \omega_n^0 \ll \omega_n^0$. Note that the eigenfrequencies ω_n cannot be measured directly, since the spectrum of spontaneous emission, i.e., the frequency differences ω_{nk} (75) corresponding to the excited modes of the atom, are measured in the spectroscopic experiments. The first term on the right-hand side of equation (88), proportional to $\varkappa$, is the same for all eigenmodes of the atom, so it does not contribute to the spectrum ω_{nk} of spontaneous emission of the atom and, consequently, cannot be determined as a result of spectroscopic measurements. The second and third terms on the right-hand side of equation (88) are different for different

eigenmodes, so they lead to a shift of the frequency ω_{nk} and, theoretically, can be measured. If, taking into account (10), one assumes that $\varkappa a_B \sim \alpha$, then the second term on the right-hand side of equation (88) will be of the order $\alpha^4 m_e c^2$, and the associated shifts of the frequency ω_{nk} will be commensurate with the Lamb shift. The third term, proportional to $\varkappa^3$, can be neglected in this case, since its value is beyond the experimental possibilities.

Let us now consider an atom in a mixed state, when both excited eigenmodes are degenerate (have the same eigenfrequency ω_n^0).

In this case, the function ψ can be written as

$$\psi = \exp(-i\omega_n t)\left(c_{m_n}\psi_{m_n} + c_{k_n}\psi_{k_n}\right) \tag{89}$$

where ψ_{m_n} and ψ_{k_n} are the eigenfunctions of equation (71), corresponding to the same eigenvalue ω_n^0 (degenerate eigenmodes of the atom); c_{k_n} and c_{m_n} are constants.

In this case, instead of equation (87), one obtains

$$-e\varphi_0(\mathbf{r}) - eu(\mathbf{r}) = \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int R|\psi|^2 dV' + \frac{1}{3}\varkappa^3 e\mathbf{rd} + \frac{1}{6}\varkappa^3 e^2 \mathbf{r}^2 + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}'^2 |\psi|^2 dV' \tag{90}$$

Using perturbation theory [32], one obtains a system of equations

$$\left(\hbar(\omega_n - \omega_n^0) + V_{m_n m_n}\right)c_{m_n} + c_{k_n} V_{m_n k_n} = 0 \tag{91}$$

$$c_{m_n} V_{k_n m_n} + \left(\hbar(\omega_n - \omega_n^0) + V_{k_n k_n}\right)c_{k_n} = 0 \tag{92}$$

where

$$V_{rs} = V_{sr}^* = \int (e\varphi_0 + eu)\psi_r^* \psi_s dV \tag{93}$$

The system of equations (91) and (92) has nonzero solutions if its determinant is equal to zero. Thus, one has the equation

$$\left(\hbar(\omega_n - \omega_n^0) + V_{m_n m_n}\right)\left(\hbar(\omega_n - \omega_n^0) + V_{k_n k_n}\right) - \left|V_{m_n k_n}\right|^2 = 0 \tag{94}$$

Equation (94) has a solution

$$\hbar(\omega_n - \omega_n^0) = -\frac{1}{2}\left(V_{k_n k_n} + V_{m_n m_n}\right) \pm \sqrt{\frac{1}{4}\left(V_{k_n k_n} - V_{m_n m_n}\right)^2 + \left|V_{m_n k_n}\right|^2} \tag{95}$$

Thus, due to the non-zero field (90), the degeneracy of the eigenmodes ψ_{m_n} and ψ_{k_n} is removed, between which a small frequency shift occurs

$$\hbar\Delta\omega_n = \sqrt{\left(V_{k_n k_n} - V_{m_n m_n}\right)^2 + 4\left|V_{m_n k_n}\right|^2} \tag{96}$$

Taking into account (90), (72), and (78), one obtains

$$V_{rs} = \varkappa e^2\delta_{rs} - \frac{1}{2}\varkappa^2 e^2 \int\int \psi_r^*(\mathbf{r})\psi_s(\mathbf{r})|\mathbf{r}-\mathbf{r}'||\psi(\mathbf{r}')|^2 dV'\,dV + \frac{1}{6}\varkappa^3(e^2\int \mathbf{r}^2\psi_r^*\psi_s dV + \delta_{rs}e^2\int \mathbf{r}^2|\psi|^2 dV - 2\mathbf{dd}_{sr}) \quad (97)$$

As indicated above, the terms containing $\varkappa^3$ can be neglected; therefore, taking into account (97), one writes the frequency shift (96) in the form

$$\hbar\Delta\omega_n = \frac{1}{2}\varkappa^2 e^2\sqrt{a^2+4|b|^2} \quad (98)$$

where

$$a = \int\int \left(\left|\psi_{k_n}(\mathbf{r})\right|^2 - \left|\psi_{m_n}(\mathbf{r})\right|^2\right)|\mathbf{r}-\mathbf{r}'||\psi(\mathbf{r}')|^2 dV'\,dV \quad (99)$$

$$b = \int\int \psi_{m_n}^*(\mathbf{r})\psi_{k_n}(\mathbf{r})|\mathbf{r}-\mathbf{r}'||\psi(\mathbf{r}')|^2 dV'\,dV \quad (100)$$

It has already been noted above that the frequency shift (98) is commensurate with the Lamb shift. This issue will be discussed in detail in the following paper of this series.

6. Nonstationary states of the atom

Let us consider an atom in a mixed state [19−23] with two simultaneously excited non-degenerate modes n and k (two-level atom):

$$\psi(\mathbf{r},t) = c_n\psi_n(\mathbf{r})\exp(-i\omega_n t) + c_k\psi_k(\mathbf{r})\exp(-i\omega_k t) \quad (101)$$

For definiteness, let us assume that $\omega_n > \omega_k$, i.e.,

$$\omega_{nk} > 0 \quad (102)$$

As we will see later, the result depends on the sign of the constant $\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)$.

6.1. The case when $\omega_{nk} < \varkappa c$

The solution of the linear Schrödinger equation (71) for the hydrogen atom yields [32]: $\frac{\omega_{nk}a_B}{c} = \frac{\alpha}{2}\left(\frac{1}{k^2}-\frac{1}{n^2}\right)$, where $n > k$; i.e., the condition $\omega_{nk} < \varkappa c$ holds for $\varkappa a_B > \frac{\alpha}{2}\left(\frac{1}{k^2}-\frac{1}{n^2}\right)$. This condition is satisfied, for example, for $\varkappa a_B = \alpha$.

In this case, equation (83) takes the form

$$\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' = \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} + e^2\int(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)\frac{1-\exp(-\varkappa R)}{R}dV' + e^2c_k^*c_n\exp(-i\omega_{nk}t)\int\psi_k^*\psi_n\frac{1-\exp\left(-\sqrt{\varkappa^2-\frac{\omega_{nk}^2}{c^2}}R\right)}{R}dV' +$$

$$e^2 c_n^* c_k \exp(i\omega_{nk}t) \int \psi_n^* \psi_k \frac{1-\exp\left(-\sqrt{\varkappa^2-\frac{\omega_{nk}^2}{c^2}}R\right)}{R} dV' - \frac{e^2}{2c^2}\omega_{nk}^2 c_k^* c_n \exp(-i\omega_{nk}t) \int R\psi_k^* \psi_n dV' -$$

$$\frac{e^2}{2c^2}\omega_{nk}^2 c_n^* c_k \exp(i\omega_{nk}t) \int R\psi_n^* \psi_k dV' \tag{103}$$

Taking into account condition (10), one considers expression (103) at points satisfying the condition $\varkappa R \ll 1$. We expand the exponents in (103) in a series in powers of R, limiting ourselves to terms of order of R^3. As a result, taking into account (72), (73), and (79), one obtains

$$\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' = \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} + \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int R(|c_n|^2|\psi_n|^2 +$$

$$|c_k|^2|\psi_k|^2)dV' - \frac{1}{2}\varkappa^2 e^2 c_k^* c_n \exp(-i\omega_{nk}t) \int R\psi_k^* \psi_n dV' - \frac{1}{2}\varkappa^2 e^2 c_n^* c_k \exp(i\omega_{nk}t) \int R\psi_n^* \psi_k dV' +$$

$$\frac{1}{6}\varkappa^3 e^2 \mathbf{r}^2 + \frac{1}{3}\varkappa^3 e\mathbf{r}(|c_n|^2\mathbf{d}_{nn} + |c_k|^2\mathbf{d}_{kk}) + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}'^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV' + \frac{1}{3}\left(\varkappa^2 -\right.$$

$$\left.\frac{\omega_{nk}^2}{c^2}\right)^{3/2} e\mathbf{r}\mathbf{d}_{nk} c_k^* c_n \exp(-i\omega_{nk}t) + \frac{1}{3}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2} e\mathbf{r}\mathbf{d}_{nk}^* c_n^* c_k \exp(i\omega_{nk}t) + \frac{1}{6}\left(\varkappa^2 -\right.$$

$$\left.\frac{\omega_{nk}^2}{c^2}\right)^{3/2} e^2 c_k^* c_n \exp(-i\omega_{nk}t) \int \mathbf{r}'^2 \psi_k^* \psi_n dV' + \frac{1}{6}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2} e^2 c_n^* c_k \exp(i\omega_{nk}t) \int \mathbf{r}'^2 \psi_n^* \psi_k dV' \tag{104}$$

For a two-level atom, equations (80) and (81) take the form

$$\ddot{\mathbf{d}}(t) = -\omega_{nk}^2 c_k^* c_n \mathbf{d}_{nk} \exp(-i\omega_{nk}t) - \omega_{nk}^2 c_n^* c_k \mathbf{d}_{nk}^* \exp(i\omega_{nk}t) \tag{105}$$

$$\dddot{\mathbf{d}}(t) = i\omega_{nk}^3 [c_k^* c_n \mathbf{d}_{nk} \exp(-i\omega_{nk}t) - c_n^* c_k \mathbf{d}_{nk}^* \exp(i\omega_{nk}t)] \tag{106}$$

Using (105), one rewrites (104) in the form

$$\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' = \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - \frac{e}{3c^3}\omega_{nk}\left(\frac{\varkappa^2 c^2}{\omega_{nk}^2} - 1\right)^{3/2} \mathbf{r}\ddot{\mathbf{d}} + \varkappa e^2 -$$

$$\frac{1}{2}\varkappa^2 e^2 \int R(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV' - \frac{1}{2}\varkappa^2 e^2 c_k^* c_n \exp(-i\omega_{nk}t) \int R\psi_k^* \psi_n dV' -$$

$$\frac{1}{2}\varkappa^2 e^2 c_n^* c_k \exp(i\omega_{nk}t) \int R\psi_n^* \psi_k dV' + \frac{1}{6}\varkappa^3 e^2 \mathbf{r}^2 + \frac{1}{3}\varkappa^3 e\mathbf{r}(|c_n|^2\mathbf{d}_{nn} + |c_k|^2\mathbf{d}_{kk}) +$$

$$\frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}'^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV' + \frac{1}{6}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2} e^2 c_k^* c_n \exp(-i\omega_{nk}t) \int \mathbf{r}'^2 \psi_k^* \psi_n dV' +$$

$$\frac{1}{6}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2} e^2 c_n^* c_k \exp(i\omega_{nk}t) \int \mathbf{r}'^2 \psi_n^* \psi_k dV' \tag{107}$$

Note that in [6], devoted to the semi-classical theory of radiation, based on intuitive arguments, the existence of a nonstationary component of the self-electric field of an atom is postulated in a form similar to the second term on the right-hand side of equation (107). In the present work, this term is a natural consequence of the self-consistent Maxwell-Pauli theory [28].

Considering the amplitudes c_n and c_k to be slow functions of time, one substitutes (101), (104), and (106) into equation (69). Taking into account (71) and (72), and discarding the rapidly oscillating terms, one obtains

$$i\hbar\dot{c}_n = \left(-i\frac{2}{3c^3}\omega_{nk}^3|\mathbf{d}_{nk}|^2|c_k|^2 - \hbar(\omega_n - \omega_n^0) + \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int|\psi_n(\mathbf{r})|^2|\mathbf{r} - \mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2 + |c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2|\psi_n|^2 dV - \frac{1}{3}\varkappa^3(|c_n|^2|\mathbf{d}_{nn}|^2 + |c_k|^2\mathbf{d}_{nn}\mathbf{d}_{kk}) + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV - \frac{1}{2}\varkappa^2 e^2|c_k|^2 \int \psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r} - \mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV - \frac{1}{3}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2}|\mathbf{d}_{nk}|^2|c_k|^2\right)c_n \quad (108)$$

$$i\hbar\dot{c}_k = \left(i\frac{2}{3c^3}\omega_{nk}^3|\mathbf{d}_{nk}|^2|c_n|^2 - \hbar(\omega_k - \omega_k^0) + \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int|\psi_k(\mathbf{r})|^2|\mathbf{r} - \mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2 + |c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2|\psi_k|^2 dV - \frac{1}{3}\varkappa^3(|c_n|^2\mathbf{d}_{nn}\mathbf{d}_{kk} + |c_k|^2|\mathbf{d}_{kk}|^2) + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV - \frac{1}{2}\varkappa^2 e^2|c_n|^2 \int \psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r} - \mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV - \frac{1}{3}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2}|\mathbf{d}_{nk}|^2|c_n|^2\right)c_k \quad (109)$$

Separating the real and imaginary terms in equations (108) and (109), one obtains

$$\hbar(\omega_n - \omega_n^0) = \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int|\psi_n(\mathbf{r})|^2|\mathbf{r} - \mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2 + |c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2|\psi_n|^2 dV - \frac{1}{3}\varkappa^3(|c_n|^2|\mathbf{d}_{nn}|^2 + |c_k|^2\mathbf{d}_{nn}\mathbf{d}_{kk}) + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV - \frac{1}{2}\varkappa^2 e^2|c_k|^2 \int \psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r} - \mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV - \frac{1}{3}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2}|\mathbf{d}_{nk}|^2|c_k|^2 \quad (110)$$

$$\hbar(\omega_k - \omega_k^0) = \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2 \int|\psi_k(\mathbf{r})|^2|\mathbf{r} - \mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2 + |c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2|\psi_k|^2 dV - \frac{1}{3}\varkappa^3(|c_n|^2\mathbf{d}_{nn}\mathbf{d}_{kk} + |c_k|^2|\mathbf{d}_{kk}|^2) + \frac{1}{6}\varkappa^3 e^2 \int \mathbf{r}^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV - \frac{1}{2}\varkappa^2 e^2|c_n|^2 \int \psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r} - \mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV - \frac{1}{3}\left(\varkappa^2 - \frac{\omega_{nk}^2}{c^2}\right)^{3/2}|\mathbf{d}_{nk}|^2|c_n|^2 \quad (111)$$

$$\dot{c}_n = -\frac{2}{3\hbar c^3}\omega_{nk}^3|\mathbf{d}_{nk}|^2|c_k|^2 c_n \quad (112)$$

$$\dot{c}_k = \frac{2}{3\hbar c^3}\omega_{nk}^3|\mathbf{d}_{nk}|^2|c_n|^2 c_k \quad (113)$$

Equations (110) and (111) describe the shifts of the eigenfrequencies of the excited modes n and k compared to those given by the linear Schrödinger equation (71). As can be seen, the shifts of the same eigenfrequencies in the mixed and pure states are different: the shifts of the eigenfrequencies (110) and (111) in the mixed state depend on the amplitudes of the excited modes. For pure states ($|c_n|^2 = 1$ and $|c_k|^2 = 0$ or $|c_n|^2 = 0$ and $|c_k|^2 = 1$), equations (110) and (111) become equation (88). Equations (112) and (113) describe the change in the amplitudes of the excited modes, i.e., the spontaneous rearrangement of the structure of the electron wave in the atom, which is traditionally called a quantum transition. As follows from equations (112) and (113), the amplitude of the eigenmode corresponding to a higher frequency decreases, while the amplitude of the eigenmode with a lower frequency increases. We can say that the electron wave (and its electric charge) flows from the mode with a higher frequency to the mode with a lower frequency [19−23]. Thus, degeneration of the mixed state of the atom occurs, in which the atom passes into a pure state corresponding to the excited mode with a lower frequency [19−23]. If in the mixed state of the atom the eigenmode n is weakly excited (which usually occurs in spontaneous emission), i.e., $|c_n|^2 \ll 1$ and $|c_k|^2 \approx 1$, then equation (112) takes the simple form

$$\dot{c}_n = -\gamma_{nk} c_n \tag{114}$$

where

$$\gamma_{nk} = \frac{2}{3\hbar c^3}\omega_{nk}^3 |\mathbf{d}_{nk}|^2 \tag{115}$$

is the damping rate of the spontaneous emission. This result coincides with the result of the theory [19−23], which does not take into account the **G** field. Note that quantum electrodynamics [33] gives the same value (115) for the damping rate of the spontaneous emission, but in the theory under consideration, this result is a natural consequence of classical field theory, without any quantization.

Spontaneous rearrangement of the structure of an atom (degeneracy of a mixed state) is associated with the loss of energy as a result of spontaneous emission, the intensity of which is equal to (17).

The intensity of electric dipole radiation is determined by relation (13), while the intensity of **G**-field radiation is determined by equations (15) and (16).

In the case under consideration, relation (82) for a two-level atom takes the form

$$u = e\int(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)\frac{\exp(-\varkappa R)}{R}dV' + ec_k^* c_n \exp(-i\omega_{nk}t)\int \psi_k^*\psi_n \frac{\exp\left(-\sqrt{\varkappa^2 - \frac{\omega_{nk}^2}{c^2}}R\right)}{R}dV' + ec_n^* c_k \exp(i\omega_{nk}t)\int \psi_n^*\psi_k \frac{\exp\left(-\sqrt{\varkappa^2 - \frac{\omega_{nk}^2}{c^2}}R\right)}{R}dV' \tag{116}$$

Using (67) and (116), it is easy to show that in the case under consideration, the **G**-field radiation flux density decreases exponentially with distance from the atom; therefore, the **G**-field radiation intensity (16)

$$I_{\mathbf{G}} = 0 \tag{117}$$

Thus, at $\omega_{nk} < \varkappa c$, the dipole radiation of the **G** field is absent. This explains why the nonlinear Schrödinger equation (69), (107) includes the term $\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}}$, which corresponds only to the electric

dipole radiation [19−23], while the term responsible for the energy dissipation associated with the **G** field is absent. That is, in the case under consideration, the rearrangement (degeneration) of the internal structure of an atom in a mixed state is associated only with the electric dipole radiation: $I = I_{\mathbf{EH}}$.

6.2. The case when $\omega_{nk} > \varkappa c$

For the hydrogen atom, this case corresponds to the condition $\varkappa a_B < \frac{\alpha}{2}\left(\frac{1}{k^2} - \frac{1}{n^2}\right)$, where $n > k$.

In this case, equation (83) takes the form

$$\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' = \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} + e^2\int(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)\frac{1-\exp(-\varkappa R)}{R}dV' + e^2c_k^*c_n\exp(-i\omega_{nk}t)\int\psi_k^*\psi_n\frac{1-\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV' + e^2c_n^*c_k\exp(i\omega_{nk}t)\int\psi_n^*\psi_k\frac{1-\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV' - \frac{e^2}{2c^2}\omega_{nk}^2c_k^*c_n\exp(-i\omega_{nk}t)\int R\psi_k^*\psi_n dV' - \frac{e^2}{2c^2}\omega_{nk}^2c_n^*c_k\exp(i\omega_{nk}t)\int R\psi_n^*\psi_k dV' \quad (118)$$

Taking into account (10), one considers expression (118) at points satisfying the condition $\varkappa R \ll 1$. One expands the exponents in (118) in a series in powers of R, limiting ourselves to terms of order of R^3.

As a result, taking into account (72), (73), and (79), one obtains

$$\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' = \frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} + \varkappa e^2 - \frac{1}{2}\varkappa^2e^2\int R(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV' - \frac{1}{2}\varkappa^2e^2c_k^*c_n\exp(-i\omega_{nk}t)\int R\psi_k^*\psi_n dV' - \frac{1}{2}\varkappa^2e^2c_n^*c_k\exp(i\omega_{nk}t)\int R\psi_n^*\psi_k dV' + \frac{1}{6}\varkappa^3e^2\mathbf{r}^2 + \frac{1}{3}\varkappa^3e\mathbf{r}(|c_n|^2\mathbf{d}_{nn} + |c_k|^2\mathbf{d}_{kk}) + \frac{1}{6}\varkappa^3e^2\int\mathbf{r}'^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV' + i\frac{1}{3}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{\frac{3}{2}}e\mathbf{r}\mathbf{d}_{nk}c_k^*c_n\exp(-i\omega_{nk}t) - i\frac{1}{3}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{\frac{3}{2}}e\mathbf{r}\mathbf{d}_{nk}^*c_n^*c_k\exp(i\omega_{nk}t) + i\frac{1}{6}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{\frac{3}{2}}e^2c_k^*c_n\exp(-i\omega_{nk}t)\int\mathbf{r}'^2\psi_k^*\psi_n dV' - i\frac{1}{6}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{3/2}e^2c_n^*c_k\exp(i\omega_{nk}t)\int\mathbf{r}'^2\psi_n^*\psi_k dV' \quad (119)$$

Using (106), one rewrites (119) in the form

$$\frac{2e}{3c^3}\mathbf{r}\ddot{\mathbf{d}} - e\varphi_0 - eu - \frac{e}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV' = \frac{e}{3c^3}\left[2 + \left(1 - \frac{\varkappa^2c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right]\mathbf{r}\ddot{\mathbf{d}} + \varkappa e^2 - \frac{1}{2}\varkappa^2e^2\int R(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV' - \frac{1}{2}\varkappa^2e^2c_k^*c_n\exp(-i\omega_{nk}t)\int R\psi_k^*\psi_n dV' -$$

$$\frac{1}{2}\varkappa^2 e^2 c_n^* c_k \exp(i\omega_{nk}t)\int R\psi_n^*\psi_k dV' + \frac{1}{6}\varkappa^3 e^2\mathbf{r}^2 + \frac{1}{3}\varkappa^3 e\mathbf{r}(|c_n|^2\mathbf{d}_{nn} + |c_k|^2\mathbf{d}_{kk}) +$$

$$\frac{1}{6}\varkappa^3 e^2\int \mathbf{r}'^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV' + i\frac{1}{6}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{\frac{3}{2}} e^2 c_k^* c_n \exp(-i\omega_{nk}t)\int \mathbf{r}'^2\psi_k^*\psi_n dV' -$$

$$i\frac{1}{6}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{3/2} e^2 c_n^* c_k \exp(i\omega_{nk}t)\int \mathbf{r}'^2\psi_n^*\psi_k dV' \tag{120}$$

It can be seen that in the case under consideration, equation (120) differs fundamentally from equation (107), which relates to the case $\omega_{nk} < \varkappa c$. First of all, this relates to the first term on the right-hand side, which is responsible for the spontaneous rearrangement of the structure of the atom in a mixed state [19−23].

As before, considering the amplitudes c_n and c_k to be slow functions of time, one substitutes (101), (120), and (106) into equation (69). Taking into account (71) and (72), and discarding the rapidly oscillating terms, one obtains

$$i\hbar\dot{c}_n = \left(-i\frac{\omega_{nk}^3}{3c^3}\left[2 + \left(1 - \frac{\varkappa^2 c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right]|\mathbf{d}_{nk}|^2|c_k|^2 - \hbar(\omega_n - \omega_n^0) + \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2\int|\psi_n(\mathbf{r})|^2|\mathbf{r} -\right.$$

$$\mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2 + |c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV + \frac{1}{6}\varkappa^3 e^2\int \mathbf{r}^2|\psi_n|^2 dV - \frac{1}{3}\varkappa^3(|c_n|^2|\mathbf{d}_{nn}|^2 +$$

$$|c_k|^2\mathbf{d}_{nn}\mathbf{d}_{kk}) + \frac{1}{6}\varkappa^3 e^2\int \mathbf{r}^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV - \frac{1}{2}\varkappa^2 e^2|c_k|^2\int\psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r} -$$

$$\left.\mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV\right)c_n \tag{121}$$

$$i\hbar\dot{c}_k = \left(i\frac{\omega_{nk}^3}{3c^3}\left[2 + \left(1 - \frac{\varkappa^2 c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right]|\mathbf{d}_{nk}|^2|c_n|^2 - \hbar(\omega_k - \omega_k^0) + \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2\int|\psi_k(\mathbf{r})|^2|\mathbf{r} -\right.$$

$$\mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2 + |c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV + \frac{1}{6}\varkappa^3 e^2\int \mathbf{r}^2|\psi_k|^2 dV - \frac{1}{3}\varkappa^3(|c_n|^2\mathbf{d}_{nn}\mathbf{d}_{kk} +$$

$$|c_k|^2|\mathbf{d}_{kk}|^2) + \frac{1}{6}\varkappa^3 e^2\int \mathbf{r}^2(|c_n|^2|\psi_n|^2 + |c_k|^2|\psi_k|^2)dV - \frac{1}{2}\varkappa^2 e^2|c_n|^2\int\psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r} -$$

$$\left.\mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV\right)c_k \tag{122}$$

Separating the real and imaginary terms in equations (121) and (122), one obtains

$$\hbar(\omega_n - \omega_n^0) = \varkappa e^2 - \frac{1}{2}\varkappa^2 e^2\int|\psi_n(\mathbf{r})|^2|\mathbf{r} - \mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2 + |c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV +$$

$$\frac{1}{6}\varkappa^3 e^2\int \mathbf{r}^2|\psi_n|^2 dV - \frac{1}{3}\varkappa^3(|c_n|^2|\mathbf{d}_{nn}|^2 + |c_k|^2\mathbf{d}_{nn}\mathbf{d}_{kk}) + \frac{1}{6}\varkappa^3 e^2\int \mathbf{r}'^2(|c_n|^2|\psi_n|^2 +$$

$$|c_k|^2|\psi_k|^2)dV' - \frac{1}{2}\varkappa^2 e^2|c_k|^2\int\psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r} - \mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV \tag{123}$$

$$\hbar(\omega_k-\omega_k^0)=\varkappa e^2-\frac{1}{2}\varkappa^2e^2\int|\psi_k(\mathbf{r})|^2|\mathbf{r}-\mathbf{r}'|(|c_n|^2|\psi_n(\mathbf{r}')|^2+|c_k|^2|\psi_k(\mathbf{r}')|^2)dV'dV+$$

$$\frac{1}{6}\varkappa^3e^2\int\mathbf{r}^2|\psi_k|^2dV-\frac{1}{3}\varkappa^3(|c_n|^2\mathbf{d}_{nn}\mathbf{d}_{kk}+|c_k|^2|\mathbf{d}_{kk}|^2)+\frac{1}{6}\varkappa^3e^2\int\mathbf{r}'^2(|c_n|^2|\psi_n|^2+$$

$$|c_k|^2|\psi_k|^2)dV'-\frac{1}{2}\varkappa^2e^2|c_n|^2\int\psi_n^*(\mathbf{r})\psi_k(\mathbf{r})|\mathbf{r}-\mathbf{r}'|\psi_k^*(\mathbf{r}')\psi_n(\mathbf{r}')dV'dV \quad (124)$$

$$\dot{c}_n=-\frac{\omega_{nk}^3}{3\hbar c^3}\left[2+\left(1-\frac{\varkappa^2c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right]|\mathbf{d}_{nk}|^2|c_k|^2c_n \quad (125)$$

$$\dot{c}_k=\frac{\omega_{nk}^3}{3\hbar c^3}\left[2+\left(1-\frac{\varkappa^2c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right]|\mathbf{d}_{nk}|^2|c_n|^2c_k \quad (126)$$

Equations (123) and (124) describe the shifts of the eigenfrequencies of the excited modes n and k compared to those given by the linear Schrödinger equation (71). They differ from the shifts of the corresponding eigenfrequencies (88) for the pure state, as well as from the shifts of eigenfrequencies (110) and (111) for the case $\omega_{nk}<\varkappa c$.

Equations (125) and (126) describe the change in the amplitudes of the excited modes, i.e., the spontaneous rearrangement of the electron wave structure in an atom in a mixed state. Equations (125) and (126) differ from the corresponding equations (112) and (113) by the damping rate of the spontaneous emission, which in the case under consideration is equal to

$$\gamma_{nk}=\frac{\omega_{nk}^3}{3\hbar c^3}\left[2+\left(1-\frac{\varkappa^2c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right]|\mathbf{d}_{nk}|^2 \quad (127)$$

In the limiting case $\varkappa=0$, one obtains [30]

$$\gamma_{nk}=\frac{\omega_{nk}^3}{\hbar c^3}|\mathbf{d}_{nk}|^2 \quad (128)$$

which is 1.5 times greater than the damping rate of the spontaneous emission (115) (see also [19−23]). The value (127) corresponds to the long-range field $\mathbf{G}$ [30].

Let us consider the physical reason for the difference between (127) and (115).

In the case under consideration, equation (82) for a two-level atom takes the form

$$u=e\int(|c_n|^2|\psi_n|^2+|c_k|^2|\psi_k|^2)\frac{\exp(-\varkappa R)}{R}dV'+$$

$$ec_k^*c_n\exp(-i\omega_{nk}t)\int\psi_k^*\psi_n\frac{\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV'+ec_n^*c_k\exp(i\omega_{nk}t)\int\psi_n^*\psi_k\frac{\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}R\right)}{R}dV' \quad (129)$$

At large distances r from the atom, one writes

$$R=|\mathbf{r}-\mathbf{r}'|=r-\mathbf{r}'\mathbf{n} \quad (130)$$

where $\mathbf{n}=\frac{\mathbf{r}}{r}$ is the unit vector in the $\mathbf{r}$ direction; $r \gg r'$.

Using (130), one writes equation (129) at large distances from the atom, neglecting exponentially small terms:

$$u=\frac{1}{r}ec_k^*c_n\exp(-i\omega_{nk}t)\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\int\psi_k^*\psi_n\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\mathbf{r}'\mathbf{n}\right)dV'+$$

$$\frac{1}{r}ec_n^*c_k\exp(i\omega_{nk}t)\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\int\psi_n^*\psi_k\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\mathbf{r}'\mathbf{n}\right)dV' \tag{131}$$

Considering $\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\mathbf{r}'\mathbf{n}\ll 1$, one expands the exponentials under the integral in a series in powers of $\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\mathbf{r}'\mathbf{n}$, limiting ourselves to first-order terms. As a result, taking into account (72) and (79), one writes (131) in the form

$$u=i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\frac{1}{r}c_k^*c_n\exp(-i\omega_{nk}t)\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}-$$

$$i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\frac{1}{r}c_n^*c_k\exp(i\omega_{nk}t)\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}^* \tag{132}$$

Using (132), one calculates in the approximation under consideration

$$\frac{\partial u}{\partial t}=\omega_{nk}\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\frac{1}{r}c_k^*c_n\exp(-i\omega_{nk}t)\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}+$$

$$\omega_{nk}\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}\frac{1}{r}c_n^*c_k\exp(i\omega_{nk}t)\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}^* \tag{133}$$

$$\nabla u=-\mathbf{n}\left(\frac{\omega_{nk}^2}{c^2}-\varkappa^2\right)\frac{1}{r}c_k^*c_n\exp(-i\omega_{nk}t)\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}-\mathbf{n}\left(\frac{\omega_{nk}^2}{c^2}-\right.$$

$$\left.\varkappa^2\right)\frac{1}{r}c_n^*c_k\exp(i\omega_{nk}t)\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}^* \tag{134}$$

Substituting (133) and (134) into (67), one obtains an expression for the energy flux density of the **G** field

$$\mathbf{J}_\mathbf{G}=\mathbf{n}\frac{1}{4\pi r^2}\omega_{nk}\left(\frac{\omega_{nk}^2}{c^2}-\varkappa^2\right)^{3/2}\left(c_k^*c_n\exp(-i\omega_{nk}t)\exp\left(i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}+\right.$$

$$\left.c_n^*c_k\exp(i\omega_{nk}t)\exp\left(-i\sqrt{\frac{\omega_{nk}^2}{c^2}-\varkappa^2}r\right)\mathbf{n}\mathbf{d}_{nk}^*\right)^2 \tag{135}$$

Expanding the brackets and averaging over time, one obtains

$$\mathbf{J_G} = \mathbf{n}\frac{1}{2\pi r^2}\omega_{nk}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{\frac{3}{2}}|c_n|^2|c_k|^2(\mathbf{nd}_{nk})(\mathbf{nd}_{nk}^*) \tag{136}$$

Substituting (136) into (16), one obtains

$$I_{\mathbf{G}} = \frac{1}{2\pi}\omega_{nk}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{\frac{3}{2}}|c_n|^2|c_k|^2\oint(\mathbf{nd}_{nk})(\mathbf{nd}_{nk}^*)d\Omega \tag{137}$$

where Ω is the solid angle.

To calculate the integral in (137), one writes

$$\mathbf{d}_{nk} = \mathbf{a} + i\mathbf{b}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the real vectors.

Then

$$|\mathbf{d}_{nk}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \tag{138}$$

$$(\mathbf{nd}_{nk})(\mathbf{nd}_{nk}^*) = (\mathbf{na})^2 + (\mathbf{nb})^2 \tag{139}$$

$$\oint(\mathbf{nd}_{nk})(\mathbf{nd}_{nk}^*)d\Omega = \oint[(\mathbf{na})^2 + (\mathbf{nb})^2]d\Omega \tag{140}$$

To calculate the integral $\oint(\mathbf{na})^2 d\Omega$, a spherical coordinate system is used, in which the *z*-axis is directed along the vector $\mathbf{a}$. In this case $(\mathbf{na}) = |\mathbf{a}|\cos\theta$ and $d\Omega = 2\pi\sin\theta\, d\theta$. Then

$$\oint(\mathbf{na})^2 d\Omega = 2\pi|\mathbf{a}|^2\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta = \frac{4\pi}{3}|\mathbf{a}|^2$$

In a similar way, one obtains

$$\oint(\mathbf{nb})^2 d\Omega = \frac{4\pi}{3}|\mathbf{b}|^2$$

Then, taking into account (138), one writes the integral (140) in the form

$$\oint(\mathbf{nd}_{nk})(\mathbf{nd}_{nk}^*)d\Omega = \oint[(\mathbf{na})^2 + (\mathbf{nb})^2]d\Omega = \frac{4\pi}{3}|\mathbf{d}_{nk}|^2 \tag{141}$$

Substituting (141) into (137), one obtains

$$I_{\mathbf{G}} = \frac{2}{3}\omega_{nk}\left(\frac{\omega_{nk}^2}{c^2} - \varkappa^2\right)^{\frac{3}{2}}|c_n|^2|c_k|^2|\mathbf{d}_{nk}|^2 \tag{142}$$

For a two-level atom, using (105) and neglecting the rapidly oscillating terms (which disappear upon averaging), one obtains

$$\ddot{\mathbf{d}}^2 = 2\omega_{nk}^4|c_n|^2|c_k|^2|\mathbf{d}_{nk}|^2 \tag{143}$$

Taking into account (143), one writes (142) in the form

$$I_{\mathbf{G}} = \frac{1}{3c^3}\left(1 - \frac{\varkappa^2c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\ddot{\mathbf{d}}^2 \tag{144}$$

For the case $\varkappa = 0$, this expression was first obtained in [30].

In the case under consideration, taking into account (13) and (144), the total intensity of spontaneous emission (17) is equal to

$$I = \frac{1}{3c^3}\left[2 + \left(1 - \frac{\varkappa^2 c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right]\ddot{\mathbf{d}}^2 \tag{145}$$

This is the energy that an atom, being in a mixed state, loses per unit time as a result of spontaneous emission. Thus, along with the spontaneous emission of electromagnetic waves, in the case under consideration there should be spontaneous emission of **G** waves, which also carry away part of the energy at a rate (144). As a result of this energy loss, a spontaneous rearrangement of the internal structure of the atom, which is traditionally called a quantum transition, occurs. In the nonlinear Schrödinger equation (69), (120), the rearrangement of the structure of the atom as a result of spontaneous emission of electromagnetic waves and **G** waves is described by the damping rate of the spontaneous emission (127), which has the same structure as (145).

Note that the constant $\varkappa$ has not yet been determined. It is only known that it must satisfy condition (10). This constant can be determined by comparing the predictions of the theory with experimental data, for example, with the damping rate of the spontaneous emission.

Thus, if $\varkappa a_B > \frac{\alpha}{2}$, then for the entire spectrum of spontaneous emission of the hydrogen atom $\omega_{nk} < \varkappa c$ and according to (115), the damping rate of the spontaneous emission

$$\gamma_{nk} = \gamma_{nk}^0 \tag{146}$$

where

$$\gamma_{nk}^0 = \frac{2}{3\hbar c^3}\omega_{nk}^3 |d_{nk}|^2 \tag{147}$$

Using the solution of the linear Schrödinger equation (71) for the hydrogen atom [32], one obtains

$$\frac{\gamma_{nk}^0}{\omega_{nk}} = \frac{1}{6}\alpha^3\left(\frac{1}{k^2} - \frac{1}{n^2}\right)^2 \left|\overline{d}_{nk}\right|^2 \tag{148}$$

where $\overline{\mathbf{d}}_{nk} = \frac{\mathbf{d}_{nk}}{ea_B}$ is the nondimensional electric dipole moment matrix.

If $\varkappa a_B < \frac{\alpha}{2}$, then at least part of the spectrum will satisfy the condition $\omega_{nk} > \varkappa c$, and for it, according to (127), the damping rate of the spontaneous emission

$$\gamma_{nk} = \gamma_{nk}^0\left[1 + \frac{1}{2}\left(1 - \frac{\varkappa^2 c^2}{\omega_{nk}^2}\right)^{\frac{3}{2}}\right] \tag{149}$$

In this case, comparing the experimental values γ_{nk} with the theoretical ones calculated using equation (149), one can estimate the value of the parameter $\varkappa a_B$.

Taking into account the above, we will calculate the damping rate of the spontaneous emission of the hydrogen atom for two mixed states. These states were chosen in such a way that the frequencies of spontaneous emission for them differed as much as possible.

As the first mixed state, we will choose

$$n = 2p_{1/2}, k = 1s_{1/2}$$

In this case, taking into account (148), (149) and the well-known solution of equation (71) for the hydrogen atom [32], one obtains

$$\frac{\varkappa c}{\omega_{nk}} = 2.13\frac{\varkappa a_B}{\alpha} \tag{150}$$

$$\overline{d}_{nk} = \left(0,0,-i\frac{2^8}{3^5\sqrt{2}}\right) \tag{151}$$

Then

$$\frac{\gamma_{nk}^0}{\omega_{nk}} \approx 0.052\alpha^3 \tag{152}$$

$$\frac{\gamma_{nk}}{\omega_{nk}} = 0.052\alpha^3\left[1+\frac{1}{2}\left(1-1.14\left(\frac{2\varkappa a_B}{\alpha}\right)^2\right)^{\frac{3}{2}}\right] \tag{153}$$

As the second mixed state, we will consider

$$n = 10p_{1/2}, k = 1s_{1/2}$$

In this case

$$\frac{\varkappa c}{\omega_{nk}} = 2.02\frac{\varkappa a_B}{\alpha} \tag{154}$$

$$\overline{d}_{nk} = (0,0,0.04i\ ) \tag{155}$$

Then

$$\frac{\gamma_{nk}^0}{\omega_{nk}} = 0.000264\alpha^3 \tag{156}$$

$$\frac{\gamma_{nk}}{\omega_{nk}} = 0.000264\alpha^3\left[1+\frac{1}{2}\left(1-1.02\left(\frac{2\varkappa a_B}{\alpha}\right)^2\right)^{\frac{3}{2}}\right] \tag{157}$$

In the case when $\omega_{nk} > \varkappa c$, the value of $\frac{\gamma_{nk}}{\omega_{nk}}$ may differ from $\frac{\gamma_{nk}^0}{\omega_{nk}}$ by up to one and a half times, which can be recorded in the experiment. It may turn out that for the entire spectrum of the hydrogen atom, $\omega_{nk} < \varkappa c$. In this case, to register this effect, it is necessary to analyze the damping rate of the spontaneous emission for other atoms emitting at higher frequencies than the hydrogen atom.

7. Conclusions

We have shown that spontaneous emission and so-called spontaneous transitions, in contrast to linear orthodox quantum mechanics and quantum electrodynamics, in the self-consistent Maxwell-Pauli theory [28,29], are a direct and natural consequence of the theory, and their explanation and description do not require additional hypotheses, such as, for example, the existence of a quantum-mechanical vacuum and its fluctuations or Zitterbewegung.

Indeed, if the atom is in a mixed state, then, as follows from Eqs. (112) and (113), as well as from Eqs. (125) and (126), the amplitude of the eigenmode corresponding to a higher frequency decreases, while the amplitude of the eigenmode with a lower frequency increases. In this case, in full accordance with classical electrodynamics, emission of electromagnetic energy occurs. Thus, a mixed state of an atom in which several eigenmodes are excited simultaneously is unstable: the electron wave (and its electric charge) spontaneously flows from the excited mode with a higher eigenfrequency to the excited mode with a lower eigenfrequency. The exceptions are mixed states with $\omega_{nk} = 0$ (i.e., with degenerate excited modes) and mixed states with $|\mathbf{d}_{nk}|^2 = 0$ (the so-called "selection rule"). Note that the eigenfrequencies (eigenvalues) of the linear Schrödinger equation (71) are negative. Therefore, the eigenfrequency of the ground mode of the hydrogen atom is lower than the frequency of any other eigenmode of the hydrogen atom. Thus, if a hydrogen atom is in a mixed state in which one of the excited eigenmodes is the ground mode, then such a mixed state of the atom is unstable and the atom will spontaneously transit to the ground state. This means that the ground state of the atom is absolutely stable.

An analysis of the complete system of equations (1)–(9) showed that the compensating short-range $\mathbf{G}$ field plays a significant role in this process, and the constant $\varkappa$ included in the inhomogeneous Klein–Gordon equation (2) determines the dynamics of spontaneous transitions. Thus, if the frequency spectrum of spontaneous radiation ω_{nk} is limited from above, $\omega_{nk} \leq \omega_{max}$ and, in addition, $\omega_{max} < \varkappa c$, then the field $\mathbf{G}$ of such an atom will not only quickly (exponentially) decay at distances greater than a_B but will also not create radiation that could be detected at macroscopic distances from the atom. In this case, macroscopic devices are not capable of directly detecting the field $\mathbf{G}$, and its existence can only be judged by indirect data.

In the case when $\omega_{nk} > \varkappa c$, despite the fact that the stationary compensating $\mathbf{G}$ field is short-range and decreases exponentially already at distances of the order of a_B from the center of the atom, $\mathbf{G}$ waves can exist that carry away energy and angular momentum [28]. Unlike the previous case, this means that in this case, although the stationary $\mathbf{G}$ field still cannot be recorded by instruments at macroscopic distances from the source, the $\mathbf{G}$ waves emitted by the atom can be detected at macroscopic distances from their source.

If the frequency spectrum of spontaneous emission ω_{nk} is limited from above, $\omega_{nk} \leq \omega_{max}$ and, in addition, $\omega_{max} > \varkappa c$, then the atom can emit $\mathbf{G}$ waves at frequencies ω_{nk} lying in the range $[\varkappa c, \omega_{max}]$ but cannot emit $\mathbf{G}$ waves at frequencies $\omega_{nk} < \varkappa c$.

Thus, although we cannot detect the stationary $\mathbf{G}$ field with macroscopic instruments due to its short-range, the results obtained give us hope that under certain conditions, we will be able to detect the wave $\mathbf{G}$ field ($\mathbf{G}$ waves) with macroscopic instruments.

To do this, it is necessary to study the emission of $\mathbf{G}$ waves and their interaction with matter (atoms), which can be done by searching for the corresponding solutions of equations (1)–(9) or their relativistic analogue [29].

This issue will be discussed in subsequent articles.

In conclusion, we note that the theory under consideration is a purely classical field theory. At the same time, we should mention other purely field models for quantum phenomena; see, e.g., [34] and references herein.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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