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Research article

Positive solutions for critical singular elliptic equations without Ambrosetti-Rabinowitz type conditions

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Abstract: We study the subcritical approximations to Li–Lin's open problem, proposed by Li and Lin (Arch Ration Mech Anal 203(3): 943-968, 2012). By applying the variational method, we obtain two positive solutions. We establish a nonexistence theorem for positive solutions. Finally, through the combination of the variational method and the sub-supersolution method, we find a global bifurcation phenomenon for positive solutions.

Keywords: Hardy-Sobolev critical exponents; positive solutions; mountain pass lemma; the least action principle; method of sub-supersolutions
Mathematics Subject Classification: 35D99, 35J15, 35J91

1. Introduction

Consider the Hardy-Sobolev's critical exponent problem

$$\begin{cases} -\Delta u = -\lambda |x|^{-s_1}|u|^{p-2}u + |x|^{-s_2}|u|^{q-2}u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, $\lambda \in \mathbb{R}$, $0 \leq s_1 < s_2 < 2$, $2^*(s) = \frac{2(N-s)}{N-2}$ for $0 \leq s \leq 2, 2 < p \leq 2^*(s_1), 2 < q \leq 2^*(s_2)$. We aim to study the existence of positive solutions of this problem when $q < 2^*(s_2)$.

In 1983, H. Brézis and L. Nirenberg [1] studied Problem (1.1) with Sobolev critical exponent where $s_1 = s_2 = 0,\ p = 2,\ q = 2^*(s_2) = 2^*(0) = \frac{2N}{N-2}$, and $\lambda \in (-\lambda_1, 0)$. Here, λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. In 2000, N. Ghoussoub and C. Yuan [2] investigated the Hardy-Sobolev critical exponent problem where $0 \in \Omega,\ s_1 = 0,\ s_2 \neq 0$ and $\lambda < 0$. Instead of $0 \in \Omega$, N. Ghoussoub and X.S. Kang considered the Hardy-Sobolev critical exponent problem where $0 \in \partial\Omega$ in their work [3].

Then, in [4], Y. Li and C.-S. Lin studied Problem (1.1) with two Hardy-Sobolev critical exponents ($0 \in \partial\Omega$, $p = 2^*(s_1) < 2^*(s_2) = q$, $\lambda \in \mathbb{R}$) and posed an open question: Does the problem have a positive solution when $\lambda > 0$ and $p > q = 2^*(s_2)$? For convenience, we refer to the two-critical Li–Lin's open problem with $2^*(s_2) = q < p = 2^*(s_1)$ and the one-critical Li–Lin's open problem with $2^*(s_2) = q < p < 2^*(s_1)$.

Our motivation is to address the open question. For more details on recent progress see [5–10] and the references therein. Specially, in 2015, G. Cerami, X. Zhong and W. Zou [7] obtained some existence results of positive solutions by the perturbation approach and the monotonicity trick. The related results are the following two theorems.

Theorem 1. (see [7], Theorem 1.5) *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a* C^1 *bounded domain such that* $0 \in \partial\Omega$. *Assume that* $\partial\Omega$ *is* C^2 *at* 0 *and* $H(0) < 0$. *Let* $0 \le s_1 < s_2 < 2$ *and* $2^*(s_2) < p \le 2^*(s_1)$. *Then there exists* $\lambda_0 > 0$ *such that Problem* (1.1) *has a positive solution for all* $\lambda \in (0, \lambda_0)$.

Theorem 2. (see [7], Theorem 1.6) *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a* C^1 *bounded domain such that* $0 \in \partial\Omega$. *Assume that* $\partial\Omega$ *is* C^2 *at* 0 *and* $H(0) < 0$. *Let* $0 \le s_1 < s_2 < 2$ *and* $2^*(s_2) < p \le 2^*(s_1) - \frac{2}{N-2}$. *Then, for almost every* $\lambda > 0$, *Problem* (1.1) *has a positive solution.*

Recently, in [10], we presented the first nonexistence result for the two-critical case of Li–Lin's open problem employing proof by contradiction, along with the Hölder inequality, the Hardy inequality, and the Young inequality. Furthermore, we obtained a second existence result for the Li–Lin's open problem with $2^*(s_1) \ge p > q = 2^*(s_2)$ based on Theorem 2. The main theorems are as follows.

Theorem 3. (see [10], Theorem 1.1) *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a domain. Assume that* $0 \le s_1 < s_2 < 2$, $p = 2^*(s_1)$ *and* $q = 2^*(s_2)$. *Then there exists* $\lambda_1 > 0$ *such that Problem* (1.1) *has no nonzero solution for all* $\lambda > \lambda_1$.

Theorem 4. (see [10], Theorem 1.4) *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a* C^1 *bounded domain such that* $0 \in \partial\Omega$. *Assume that* $\partial\Omega$ *is* C^2*at* 0 *and* $H(0) < 0$. *Let* $0 \le s_1 < s_2 < 2$, $q = 2^*(s_2)$, $2^*(s_1) - \frac{N(s_2-s_1)}{(N-2)(N+1-s_2)} < p \le 2^*(s_1)$ *and* $2^*(s_1) - \frac{s_2-s_1}{N-2} \le p$. *Let* $\lambda_* = \sup\{\lambda \in \mathbb{R} \mid$ *Problem* (1.1) *has a positive solution*$\}$. *Then* $\lambda_* > 0$ *and Problem* (1.1) *has at least a positive solution for all* $\lambda \in (0, \lambda_*)$.

Clearly, the present results only deal with special cases of Li–Lin's open problem and are far from giving a full solution. The main difficulty of this problem is that it's impossible to obtain the boundedness of the (PS) sequences for the energy functional.

A natural question is: What will happen if we exchange the critical property of p and q in the one-critical Li–Lin's open problem, that is, $p = 2^*(s_1), 2 < q < 2^*(s_2)$? This question is the same as replacing $q = 2^*(s_2)$ with $q < 2^*(s_2)$ in the two-critical Li–Lin's open problem. In fact, obtaining the boundedness of the (PS) sequences for the energy functional is still a challenge. However, we have proved a new inequality to overcome this difficulty.

In this paper, we study more general questions, that is, $2 < p \le 2^*(s_1), 2 < q < 2^*(s_2)$. It is noteworthy that for small λ, we can obtain two positive solutions, while for large λ, there are no positive solutions. The main results are the following theorems.

Theorem 1.1. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with* $0 \in \partial\Omega$, $\lambda > 0$, $0 \le s_1 < s_2 < 2$, $2 < p \le 2^*(s_1)$ *and* $2 < q < \frac{2-s_2}{2-s_1}p + \frac{2s_2-2s_1}{2-s_1}$. *Then there exists* $\lambda_0 > 0$ *such that Problem* (1.1) *has at least two positive solutions for all* $\lambda \in (0, \lambda_0)$.

Corollary 1.2. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with* $0 \in \partial\Omega$, $\lambda > 0, 0 \le s_1 < s_2 < 2$, $p = 2^*(s_1)$ *and* $2 < q < 2^*(s_2)$. *Then there exists* $\lambda_0 > 0$ *such that Problem* (1.1) *has at least two positive solutions for all* $\lambda \in (0, \lambda_0)$.

Theorem 1.3. *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with* $0 \in \partial\Omega$, $\lambda > 0$, $0 \le s_1 < s_2 < 2$, $2 < p \le 2^*(s_1)$ *and* $2 < q < \frac{N+2-2s_2}{N+2-2s_1}p + \frac{2s_2-2s_1}{N+2-2s_1}$. *Then there exists* $\lambda_* > 0$ *such that Problem* (1.1) *has no positive solution for all* $\lambda > \lambda_*$, *and has at least one positive solution for all* $\lambda \in (0, \lambda_*]$.

Corollary 1.4. *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with* $0 \in \partial\Omega$, $\lambda > 0$, $0 \le s_1 < s_2 < 2$, $p = 2^*(s_1)$ *and* $2 < q < 2^*(s_2)$. *Then there exists* $\lambda_* > 0$ *such that Problem* (1.1) *has no positive solution for all* $\lambda > \lambda_*$, *and has at least one positive solution for all* $\lambda \in (0, \lambda_*]$.

Remark 1.5. *This question is related to the subcritical approximations of the two-critical Li–Lin's open problem.*

Remark 1.6. *Theorem 1.3 represents a global bifurcation for positive solutions. Moreover, its proof is carried out using the variational method combined with the method of sub-supersolutions.*

2. Preliminaries

We introduce the work space

$$E = H_0^1(\Omega)$$

with scalar product and norm given by

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v dx \qquad \text{and} \qquad \|u\| = (u, u)^{\frac{1}{2}}.$$

It is well-known that the solutions of problem (1.1) are precisely the critical points of the energy functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I_\lambda(u) \quad = \quad \frac{1}{2}\|u\|^2 + \frac{\lambda}{p}\int_\Omega |x|^{-s_1}|u|^p dx - \frac{1}{q}\int_\Omega |x|^{-s_2}|u|^q dx.$$

It is easy to see that I_λ is well-defined and $I_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$. Then, for any $u, v \in H_0^1(\Omega)$,

$$\langle I'_\lambda(u),\ v\rangle \triangleq \lim_{t\to 0}\frac{1}{t}[I_\lambda(u + tv) - I_\lambda(u)] = (u, v) + \lambda\int_\Omega |x|^{-s_1}|u|^{p-2}uvdx - \int_\Omega |x|^{-s_2}|u|^{q-2}uvdx$$

where $I'_\lambda(u)$ is the Gâteaux derivative of $I_\lambda(u)$.

Proposition 2.1. *Suppose that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with* $0 \in \partial\Omega$, $0 \le s_1 < s_2 < 2$, $2 < p \le 2^*(s_1)$ *and* $2 < q < \frac{2-s_2}{2-s_1}p + \frac{2s_2-2s_1}{2-s_1}$. *Then there exist three positive constants* $\gamma_1, \gamma_2, \gamma_3$ *with* $\gamma_1+\gamma_2+\gamma_3 = 1$ *such that*

$$\int_\Omega |x|^{-s_2}|u|^q dx \le \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(\int_\Omega |x|^{-2}|u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}$$

for every $u \in H_0^1(\Omega)$.

Proof Let

$$\gamma_1 = \frac{2(q-s_2)}{2(p-s_1)},$$
$$\gamma_2 = \frac{ps_2-qs_1}{2(p-s_1)},$$
$$\gamma_3 = \frac{(2-s_2)p-(2-s_1)q+2(s_2-s_1)}{2(p-s_1)}.$$

Then we have $\gamma_1, \gamma_2, \gamma_3 > 0$ and

$$\begin{aligned} p\gamma_1 + 2\gamma_2 &= q, \\ s_1\gamma_1 + 2\gamma_2 &= s_2, \\ \gamma_1 + \gamma_2 + \gamma_3 &= 1. \end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned} \int_\Omega |x|^{-s_2}|u|^q dx &= \int_\Omega |x|^{-s_1\gamma_1}|u|^{p\gamma_1} \cdot |x|^{-2\gamma_2}|u|^{2\gamma_2} \cdot 1 dx \\ &\le \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1} \left(\int_\Omega |x|^{-2}|u|^2 dx\right)^{\gamma_2} |\Omega|^{\gamma_3} \end{aligned}$$

for every $u \in H_0^1(\Omega)$. This completes the proof of the proposition. □

Lemma 2.2. *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $0 \in \partial\Omega$, $\lambda > 0$, $0 \le s_1 < s_2 < 2$, $2 < p \le 2^*(s_1)$ and $2 < q < \frac{2-s_2}{2-s_1}p + \frac{2s_2-2s_1}{2-s_1}$. Then there exists $\lambda_1 > 0$ such that Problem* (1.1) *has no nonzero solution for every $\lambda > \lambda_1$.*

Proof Suppose that u is a nonzero solution of Problem (1.1). Then one has $I'_\lambda(u) = 0$. Hence, $\langle I'_\lambda(u), u\rangle = 0$, that is,

$$\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega |x|^{-s_1}|u|^p dx = \int_\Omega |x|^{-s_2}|u|^q dx, \tag{2.1}$$

which implies that

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx &\le \int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega |x|^{-s_1}|u|^p dx \\ &= \int_\Omega |x|^{-s_2}|u|^q dx \\ &\le C\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{q}{2}} \end{aligned}$$

for some constant $C > 0$ according to the Hardy-Sobolev inequality. It follows that

$$1 \le C^{\frac{2}{q-2}} \int_\Omega |\nabla u|^2 dx.$$

By Proposition 2.1 and the Hardy inequality (see [11], Theorem 4.1), we have

$$\int_\Omega |x|^{-s_2}|u|^q dx \le \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1} \left(\int_\Omega |x|^{-2}|u|^2 dx\right)^{\gamma_2} |\Omega|^{\gamma_3}$$

$$
\begin{aligned}
&\leq \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(C_N\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}\\
&\leq \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(C_N\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}C^{\frac{2\gamma_3}{q-2}}\left(\int_\Omega |\nabla u|^2 dx\right)^{\gamma_3}\\
&= \left((C_N^{\gamma_2}|\Omega|^{\gamma_3}C^{\frac{2\gamma_3}{q-2}})^{\frac{1}{\gamma_1}}\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2+\gamma_3},
\end{aligned}
$$

where $C_N = \left(\frac{2}{N-2}\right)^2$. It follows from the Young inequality that

$$
\begin{aligned}
\int_\Omega |\nabla u|^2 dx + \lambda\int_\Omega |x|^{-s_1}|u|^p dx &= \int_\Omega |x|^{-s_2}|u|^q dx\\
&\leq \gamma_1(C_N^{\gamma_2}|\Omega|^{\gamma_3}C^{\frac{2\gamma_3}{q-2}})^{\frac{1}{\gamma_1}}\int_\Omega |x|^{-s_1}|u|^p dx\\
&\quad +(\gamma_2+\gamma_3)\int_\Omega |\nabla u|^2 dx\\
&\leq \gamma_1(C_N^{\gamma_2}|\Omega|^{\gamma_3}C^{\frac{2\gamma_3}{q-2}})^{\frac{1}{\gamma_1}}\int_\Omega |x|^{-s_1}|u|^p dx\\
&\quad +\int_\Omega |\nabla u|^2 dx,
\end{aligned}
$$

which implies that

$$
\lambda \leq \gamma_1(C_N^{\gamma_2}|\Omega|^{\gamma_3}C^{\frac{2\gamma_3}{q-2}})^{\frac{1}{\gamma_1}}
$$

by (2.1). Let $\lambda_1 = \gamma_1(C_N^{\gamma_2}|\Omega|^{\gamma_3}C^{\frac{2\gamma_3}{q-2}})^{\frac{1}{\gamma_1}}$, then Problem (1.1) has no nonzero solution for every $\lambda > \lambda_1$. □

Lemma 2.3. *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $0 \in \partial\Omega$, $0 \leq s_1 < s_2 < 2$, $2 < p \leq 2^*(s_1)$ and $2 < q < \frac{2-s_2}{2-s_1}p + \frac{2s_2-2s_1}{2-s_1}$. Then the functional I_λ is coercive, i.e., $I_\lambda(u) \to +\infty$ as $\|u\| \to \infty$. Moreover, the functional I_λ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$. Specifically, if $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$, then $\{u_n\}$ has a convergent subsequence.*

Proof By Proposition 2.1, the Hardy inequality and the Young inequality, we have

$$
\begin{aligned}
\int_\Omega |x|^{-s_2}|u|^q dx &\leq \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(\int_\Omega |x|^{-2}|u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}\\
&\leq \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(C_N\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}\\
&= \left(p^{-1}\lambda\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(A\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2}\\
&= \left(p^{-1}\lambda\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left\{\left(A\int_\Omega |\nabla u|^2 dx\right)^{\frac{\gamma_2}{1-\gamma_1}}\right\}^{1-\gamma_1}\\
&\leq \gamma_1 p^{-1}\lambda\int_\Omega |x|^{-s_1}|u|^p dx + (1-\gamma_1)A^{\frac{\gamma_2}{1-\gamma_1}}\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{\gamma_2}{1-\gamma_1}}
\end{aligned}
$$

$$\leq \quad p^{-1}\lambda \int_{\Omega} |x|^{-s_1}|u|^p dx + A^{\frac{\gamma_2}{1-\gamma_1}} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{\gamma_2}{1-\gamma_1}},$$

where $C_N = \left(\frac{2}{N-2}\right)^2$ and $A = C_N(p\lambda^{-1})^{\frac{\gamma_1}{\gamma_2}}|\Omega|^{\frac{\gamma_3}{\gamma_2}}$. It follows that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx + \frac{\lambda}{p}\int_{\Omega} |x|^{-s_1}|u|^p dx - \frac{1}{q}\int_{\Omega} |x|^{-s_2}|u|^q dx \\ &\geq \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx - A^{\frac{\gamma_2}{1-\gamma_1}} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{\gamma_2}{1-\gamma_1}} \end{aligned}$$

on $H_0^1(\Omega)$, which implies that the functional I_λ is coercive due to $0 < \frac{\gamma_2}{1-\gamma_1} < 1$.

Suppose that $\{u_n\}$ is a $(PS)_c$ sequence for some $c \in \mathbb{R}^N$, that is, $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$. Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$ by the coercivity of I_λ. Up to a subsequence, there is $u_0 \in H_0^1(\Omega)$ such that, as $n \to \infty$,

$$\begin{aligned} &u_n \rightharpoonup u_0 && \text{in } H_0^1(\Omega), \\ &u_n \to u_0 && \text{in } L^q(\Omega; |x|^{-s_2}) \text{ for } q \in [2, 2^*(s_2)), \\ &u_n(x) \to u_0(x) && a.e. \text{ in } \Omega. \end{aligned}$$

By the definition of I'_λ we have

$$\begin{aligned} \langle I'_\lambda(u_n) - I'_\lambda(u_0),\ u_n - u_0 \rangle &= (u_n - u_0, u_n - u_0) \\ &\quad + \lambda \int_{\Omega} |x|^{-s_1}(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0) dx \\ &\quad - \int_{\Omega} |x|^{-s_2}(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)(u_n - u_0) dx \\ &\geq \|u_n - u_0\|^2 - \int_{\Omega} |x|^{-s_2}(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)(u_n - u_0) dx, \end{aligned}$$

which implies that $u_n \to u_0$ in $H_0^1(\Omega)$ as $n \to \infty$ by

$$\begin{aligned} |\langle I'_\lambda(u_n) - I'_\lambda(u_0),\ u_n - u_0 \rangle| &\leq \|I'_\lambda(u_n)\| \|u_n - u_0\| + |\langle I'_\lambda(u_0),\ u_n - u_0 \rangle| \\ &\leq C_1 \|I'_\lambda(u_n)\| + |\langle I'_\lambda(u_0),\ u_n - u_0 \rangle| \\ &\to 0 \end{aligned}$$

and

$$\begin{aligned} &\left| \int_{\Omega} |x|^{-s_2}(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)(u_n - u_0) dx \right| \\ \leq\ & \int_{\Omega} |x|^{-s_2}(|u_n|^{q-1} + |u_0|^{q-1})|u_n - u_0| dx \\ \leq\ & \int_{\Omega} |x|^{-s_2}|u_n|^{q-1}|u_n - u_0| dx + \int_{\Omega} |x|^{-s_2}|u_0|^{q-1}|u_n - u_0| dx \\ \leq\ & \left(\int_{\Omega} |x|^{-s_2}|u_n|^q dx \right)^{\frac{q-1}{q}} \left(\int_{\Omega} |x|^{-s_2}|u_n - u_0|^q dx \right)^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&+\left(\int_\Omega |x|^{-s_2}|u_0|^q dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |x|^{-s_2}|u_n-u_0|^q dx\right)^{\frac{1}{q}}\\
\le\ & C_2\left(\int_\Omega |x|^{-s_2}|u_n-u_0|^q dx\right)^{\frac{1}{q}}\\
\to\ & 0,
\end{aligned}
$$

where C_1 and C_2 are some positive constants. □

3. Proof of Theorems 1.1 and 1.3

3.1. Proof of Theorem 1.1

Proof At the beginning, we define

$$
\lambda_0 \triangleq \sup_{u\neq 0}\left\{\frac{-\frac{1}{2}\int_\Omega |\nabla u|^2 dx+\frac{1}{q}\int_\Omega |x|^{-s_2}|u|^q dx}{\frac{1}{p}\int_\Omega |x|^{-s_1}|u|^p dx}\right\}.
$$

Our goal is to prove that $0<\lambda_0<+\infty$. We choose $u_0\in H_0^1(\Omega)\setminus\{0\}$. Since $q>2$, there exists $t_0>0$ such that

$$
-\frac{1}{2}\int_\Omega |\nabla(t_0u_0)|^2 dx+\frac{1}{q}\int_\Omega |x|^{-s_2}|t_0u_0|^q dx>0.
$$

According to the definition of λ_0, one has $\lambda_0>0$.

Now we prove $\lambda_0<+\infty$. On the one hand, by the Hardy-Sobolev inequality, since $2<q$, there is a positive constant r_0 such that

$$
\frac{-\frac{1}{2}\int_\Omega |\nabla u|^2 dx+\frac{1}{q}\int_\Omega |x|^{-s_2}|u|^q dx}{\frac{1}{p}\int_\Omega |x|^{-s_1}|u|^p dx}\le 0
$$

for all $u\in H_0^1(\Omega)\setminus\{0\}$ with $\|u\|\le r_0$. On the other hand, for $\|u\|>r_0$, we have

$$
\begin{aligned}
\int_\Omega |x|^{-s_2}|u|^q dx &\le \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(\int_\Omega |x|^{-2}|u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}\\
&\le \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(C_N\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}\\
&\le \left(\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(C_N\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2}|\Omega|^{\gamma_3}r_0^{-2\gamma_3}\left(\int_\Omega |\nabla u|^2 dx\right)^{\gamma_3}\\
&= \left((C_N^{\gamma_2}|\Omega|^{\gamma_3}r_0^{-2\gamma_3})^{\frac{1}{\gamma_1}}\int_\Omega |x|^{-s_1}|u|^p dx\right)^{\gamma_1}\left(\int_\Omega |\nabla u|^2 dx\right)^{\gamma_2+\gamma_3}
\end{aligned}
$$

by Proposition 2.1 and the Hardy inequality. Moreover, it follows from the Young inequality that

$$
-\frac{1}{2}\int_\Omega |\nabla u|^2 dx+\frac{1}{q}\int_\Omega |x|^{-s_2}|u|^q dx \le \gamma_1(C_N^{\gamma_2}|\Omega|^{\gamma_3}r_0^{-2\gamma_3})^{\frac{1}{\gamma_1}}\int_\Omega |x|^{-s_1}|u|^p dx
$$

$$+\left(\frac{\gamma_2+\gamma_3}{q}-\frac{1}{2}\right)\int_\Omega |\nabla u|^2dx$$
$$\leq \gamma_1(C_N^{\gamma_2}|\Omega|^{\gamma_3}r_0^{-2\gamma_3})^{\frac{1}{\gamma_1}}\int_\Omega |x|^{-s_1}|u|^p dx.$$

Therefore,

$$\frac{-\frac{1}{2}\int_\Omega |\nabla u|^2dx+\frac{1}{q}\int_\Omega |x|^{-s_2}|u|^q dx}{\frac{1}{p}\int_\Omega |x|^{-s_1}|u|^p dx} \leq p\gamma_1(C_N^{\gamma_2}|\Omega|^{\gamma_3}C^{\frac{2\gamma_3}{q-2}})^{\frac{1}{\gamma_1}}$$

for $\|u\| > r_0$. By the definition of λ_0, we can see that $\lambda_0 < +\infty$.

In conclusion, we have $0 < \lambda_0 < +\infty$.

Next, we define

$$m_\lambda \triangleq \inf\{I_\lambda(u) \mid u \in H_0^1(\Omega)\}$$

and want to prove that $-\infty < m_\lambda < 0$ for every $\lambda \in (0, \lambda_0)$. By the definition of λ_0, there exists $u_\lambda \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\lambda < \frac{-\frac{1}{2}\int_\Omega |\nabla u_\lambda|^2dx+\frac{1}{q}\int_\Omega |x|^{-s_2}|u_\lambda|^q dx}{\frac{1}{p}\int_\Omega |x|^{-s_1}|u_\lambda|^p dx},$$

which implies that

$$I_\lambda(u_\lambda)=\frac{1}{2}\int_\Omega |\nabla u_\lambda|^2dx+\frac{\lambda}{p}\int_\Omega |x|^{-s_1}|u_\lambda|^p dx-\frac{1}{q}\int_\Omega |x|^{-s_2}|u_\lambda|^q dx<0.$$

Thus, $m_\lambda < 0$. Furthermore, it follows from Lemma 2.3 and the boundedness of functional I_λ that $m_\lambda > -\infty$. Therefore, we have proven that $-\infty < m_\lambda < 0$.

Now we prove the existence of a positive solution, which is the local minimum point of the functional I_λ. Note that the functional I_λ is weakly lower semi-continuous since

$$I_1(u) \triangleq \frac{1}{2}\int_\Omega |\nabla u|^2dx+\frac{\lambda}{p}\int_\Omega |x|^{-s_1}|u|^p dx$$

is convex and continuous, and

$$I_2(u) \triangleq -\frac{1}{q}\int_\Omega |x|^{-s_2}|u|^q dx$$

is weakly continuous. By the least action principle (see Theorem 1.1 in [12]), I_λ has a minimum point w_λ such that $I_\lambda(w_\lambda) = m_\lambda$. Due to $m_\lambda < 0$, one has $w_\lambda \neq 0$. Since $I_\lambda(|w_\lambda|) = I_\lambda(w_\lambda) = m_\lambda$, we may assume that $w_\lambda \geq 0$. Hence, w_λ is a nonzero nonnegative solution of Problem (1.1). It follows from the strong maximum principle (see [13]) that w_λ is a positive solution of Problem (1.1).

Finally, we consider another positive solution which is the mountain pass point of the functional I_λ. By the Hardy-Sobolev inequality, we have

$$\int_\Omega |x|^{-s_2}|u|^q dx \leq C\|u\|^q$$

for all $u \in H_0^1(\Omega)$ and some constant $C > 0$. It follows that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{p}\int_\Omega |x|^{-s_1}|u|^p dx - \frac{1}{q}\int_\Omega |x|^{-s_2}|u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^q \end{aligned}$$

on $H_0^1(\Omega)$, which implies that $I_\lambda(u) \geq \frac{1}{4}\rho^2$ for $\|u\| = \rho$ with $0 < \rho < \min\{\|w_\lambda\|, (4C)^{-\frac{1}{q-2}}\}$. Note that $I_\lambda(w_\lambda) = m_\lambda < 0$. Hence, the functional I_λ has a mountain pass geometry structure. Then we define the minimax value

$$c_\lambda := \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where

$$\Gamma_\lambda := \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = w_\lambda\right\}.$$

According to Lemma 2.3 and the mountain pass lemma (see [14]), I_λ has a mountain pass point v_λ such that $I_\lambda(v_\lambda) = c_\lambda$. Since $c_\lambda > 0$, we know that $v_\lambda \neq 0$. Also, because $I_\lambda(|v_\lambda|) = I_\lambda(v_\lambda) = c_\lambda$, we can assume that $v_\lambda \geq 0$. Consequently, v_λ is a nonzero nonnegative solution of Problem (1.1). By the strong maximum principle, v_λ is a positive solution of Problem (1.1). Moreover, since $I_\lambda(v_\lambda) = c_\lambda > 0 > m_\lambda = I_\lambda(w_\lambda)$, we have $v_\lambda \neq w_\lambda$.

In conclusion, for all $\lambda \in (0, \lambda_0)$, Problem (1.1) has at least two positive solutions v_λ and w_λ. □

3.2. Proof of Theorem 1.3

In this subsection, we will prove Theorem 1.3 using the method of sub-supersolutions and the variational method. Now, we recall the sub-supersolution method in [15].

Definition 3.1. (see [15], P2430, Definition 1.1) *A function u is an L^1-solution of*

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, if

(*i*) $u \in L^1(\Omega)$;

(*ii*) $f(\cdot, u)\rho_0 \in L^1(\Omega)$;

(*iii*)

$$-\int_\Omega u\Delta\varphi dx = \int_\Omega f(x,u)\varphi dx \ \ \forall\varphi \in C_0^2(\overline{\Omega}). \tag{3.2}$$

Here, $\rho_0(x) = d(x, \partial\Omega)$, $\forall x \in \Omega$, and $C_0^2(\overline{\Omega}) = \{\varphi \in C^2(\overline{\Omega});\ \varphi = 0$ on $\partial\Omega\}$.

We also consider L^1-*subsolutions* and L^1-*supersolutions* in analogy with this definition. For instance, u is an L^1-*subsolution* of Problem (3.1) if u satisfies (i)-(iii) with "≤" instead of "=" in (3.2). We will systematically omit the term "L^1" and simply say that u is a solution of Problem (3.1), which means that u satisfies (3.2); a similar convention applies to subsolutions and supersolutions.

Lemma 3.2. (see [15], P2436, Corollary 5.3) *Let* $v_1, v_2 \in L^1(\Omega)$ *be a sub and a supersolution of Problem* (3.1)*, respectively. Assume that* $v_1 \le v_2$ *a.e. and*

$$f(\cdot, v) \in L^{\frac{2N}{N+2}}(\Omega) \quad \textit{for every } v \in L^1(\Omega) \textit{ such that } v_1 \le v \le v_2 \textit{ a.e.} \tag{3.3}$$

Then, Problem (3.1) *has a solution* $u \in H_0^1(\Omega)$ *such that* $v_1 \le u \le v_2$ *a.e.*

Proof of Theorem 1.3 We define

$$\lambda_* = \sup\{\lambda \in \mathbb{R} \mid \text{Problem (1.1) has a positive solution}\}.$$

From Theorem 1.1 and Lemma 2.2, we obtain $\lambda_* \in (0, +\infty)$. Hence, Problem (1.1) has no positive solution for all $\lambda > \lambda_*$.

By the definition of λ_*, for every $\lambda \in (0, \lambda_*)$, there exists $\mu \in (\lambda, \lambda_*)$ such that Problem (1.1) with $\lambda = \mu$ has a positive solution u_μ.

Let $v_1 = u_\mu$, $v_2(x) \triangleq M|x|^{-\alpha}$, $\alpha \triangleq \frac{s_2 - s_1}{p-q}$ and

$$M \triangleq \max\left\{\lambda^{-\frac{1}{p-q}}, \|u_\mu\|_\infty \sup\left\{|x|^\alpha \Big| x \in \overline{\Omega}\right\}\right\}.$$

We have $\alpha \le N-2$, which follows from $q < \frac{N+2-2s_2}{N+2-2s_1}p + \frac{2s_2-2s_1}{N+2-2s_1}$.

In a way similar to the proof in [10], one obtains that v_1 is a subsolution of Problem (1.1), v_2 is a supersolution of Problem (1.1) and $v_2(x) \ge v_1(x)$ for a.e. $x \in \overline{\Omega}$.

By $q < \frac{N+2-2s_2}{N+2-2s_1}p + \frac{2s_2-2s_1}{N+2-2s_1}$, we have $[\alpha(p-1)+s_1]\frac{2N}{N+2} = [\alpha(q-1)+s_2]\frac{2N}{N+2} < N$, which implies that (3.3) holds. Hence, Problem (1.1) has at least one positive solution for all $\lambda \in (0, \lambda_*)$ by Lemma 3.2.

Next, we consider the case $\lambda = \lambda_*$. For an integer $n > \frac{1}{\lambda_*}$, there exists a positive $u_n \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u_n = -(\lambda_* - 1/n)|x|^{-s_1}|u_n|^{p-2}u_n + |x|^{-s_2}|u_n|^{q-2}u_n & \text{in } \Omega, \\ u_n(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

For any $v \in H_0^1(\Omega)$, we have

$$\begin{aligned} |(u_n, v)| &\le (\lambda_* - 1/n)\int_\Omega |x|^{-s_1}|u_n|^{p-1}|v|dx + \int_\Omega |x|^{-s_2}|u_n|^{q-1}|v|dx \\ &\le \lambda_*\left(\int_\Omega |x|^{\frac{-2Ns_1}{N+2}}|u_n|^{\frac{2N(p-1)}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_\Omega |v|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}} \\ &\quad + \left(\int_\Omega |x|^{\frac{-2Ns_2}{N+2}}|u_n|^{\frac{2N(q-1)}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_\Omega |v|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}} \\ &\le (\lambda_* M^{\frac{2N(p-1)}{N+2}} + M^{\frac{2N(q-1)}{N+2}})\left(\int_\Omega |x|^{\frac{-2Ns_1}{N+2}}|x|^{\frac{-\alpha 2N(p-1)}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_\Omega |v|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}} \\ &\le C\|v\| \end{aligned}$$

for some positive constant C. So $\|u_n\| \le C$, and $I_{\lambda_*}(u_n)$ is bounded. Without loss of generality, assume that $I_{\lambda_*}(u_n) \to c$ as $n \to \infty$ for some $c \in \mathbb{R}$. From (3.4), we get $I'_{\lambda_*}(u_n) = 1/n|x|^{-s_1}|u_n|^{p-2}u_n \to 0$ as $n \to \infty$. Thus, (u_n) is a $(PS)_c$ sequence of I_{λ_*}. According to Lemma 2.3, (u_n) has a convergent subsequence. We can assume that $u_n \to u_0$ as $n \to \infty$ for some $u_0 \in H_0^1(\Omega)$. Moreover, u_0 is a nonnegative nonzero solution of Problem (1.1) with $\lambda = \lambda_*$. By the strong maximum principle, u_0 is a positive solution of Problem (1.1) with $\lambda = \lambda_*$, which completes our proof. □

Author contributions

Zhi-Yun Tang: Conceptualization, Writing–Original Draft, Writing–Review & Editing; Xianhua Tang: Supervision, Writing–Review & Editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.*, **36** (1983), 437–477. https://doi.org/10.1002/cpa.3160360405
2. N. Ghoussoub, C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, *Trans. Amer. Math. Soc.*, **352** (2000), 5703–5743. https://doi.org/10.1090/S0002-9947-00-02560-5
3. N. Ghoussoub, X. S. Kang, Hardy-Sobolev critical elliptic equations with boundary singularities, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **21** (2004), 767–793. https://doi.org/10.1016/j.anihpc.2003.07.002
4. Y.Y. Li, C. S. Lin, A nonlinear elliptic PDE and two Sobolev-Hardy critical exponents, *Arch. Ration. Mech. Anal.*, **203** (2012), 943–968. https://doi.org/10.1007/s00205-011-0467-2
5. T. Jin, Symmetry and nonexistence of positive solutions of elliptic equations and systems with Hardy terms, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **28** (2011), 965–981. https://doi.org/10.1016/j.anihpc.2011.07.003
6. S. Yan, J. Yang, Infinitely many solutions for an elliptic problem involving critical Sobolev and Hardy-Sobolev exponents, *Calc. Var. Partial Differential Equations*, **48** (2013), 587–610. https://doi.org/10.1007/s00526-012-0563-7
7. G. Cerami, X. Zhong, W. Zou, On some nonlinear elliptic PDEs with Sobolev-Hardy critical exponents and a Li-Lin open problem, *Calc. Var. Partial Differential Equations*, **54** (2015), 1793–1829. https://doi.org/10.1007/s00526-015-0844-z

8. X. Zhong, W. Zou, A nonlinear elliptic PDE with multiple Hardy-Sobolev critical exponents in $\mathbb{R}^N$, *J. Differential Equations*, **292** (2021), 354–387. https://doi.org/10.1016/j.jde.2021.05.027
9. C. Wang, J. Su, The ground state solutions of Hénon equation with upper weighted critical exponents, *J. Differential Equations*, **302** (2021), 444–473. https://doi.org/10.1016/j.jde.2021.09.007
10. Z. Y. Tang, X. H. Tang, On Li-Lin's open problem, *J. Differential Equations*, **435** (2025), 113244. https://doi.org/10.1016/j.jde.2025.113244
11. H. Brezis, J.L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, *Rev. Mat. Univ. Complut. Madrid*, **10** (1997), 443–469. https://doi.org/10.5209/rev_rema.1997.v10.n2.17459
12. J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, volume 74 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989. https://doi.org/10.1007/978-1-4757-2061-7
13. D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
14. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis*, **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7
15. M. Montenegro, A. C. Ponce, The sub-supersolution method for weak solutions, *Proc. Amer. Math. Soc.*, **136** (2008), 2429–2438. https://doi.org/10.1090/S0002-9939-08-09231-9