



Research article

Uniform bound of the highest-order energy for three dimensional inhomogeneous incompressible elastodynamics

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Abstract: We are concerned with the time growth of the highest-order energy of three-dimensional inhomogeneous incompressible isotropic elastodynamics. Utilizing Klainerman's generalized energy method, refined weighted estimates, and the Keel-Smith-Sogge estimate [J. Anal. Math., 87: 265–279, 2002], it is justified that the highest-order generalized energy is uniformly bounded for all time.

Keywords: inhomogeneous incompressible elastodynamics; uniform bound; highest-order energy; KSS estimate; vector field theory

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1. Introduction

The motion dynamics of incompressible isotropic elastodynamics is characterized as a wave system in Lagrangian coordinates, which inherently satisfies the null condition. Based on this structure, a series of studies have been conducted to establish the global well-posedness of classical solutions to this system; see [1, 2]. However, these studies reveal a certain time growth for the highest-order generalized energy. In this paper, we investigate the time growth of the Sobolev norm for classical solutions to three-dimensional inhomogeneous incompressible isotropic elastodynamics with small initial perturbation and establish the uniform bound for the highest-order energy.

Before presenting the main result of this paper, we briefly review related known results. For three-dimensional elastic waves, John [3] proved that the genuine nonlinearity condition leads to singularity formation even for arbitrarily small spherically symmetric displacement. We also refer readers to [4] regarding large displacement singularity. The existence of almost global solutions was established in [5, 6] for three-dimensional quasilinear wave equations with sufficiently small initial data. Significant contributions toward global existence were independently made by Sideris [7, 8] and Agemi [9] under the assumption that nonlinearity satisfies the null condition in three dimensions.

In terms of three-dimensional incompressible elastodynamics, the only waves presented in the isotropic systems are shear waves, which are linearly degenerate. The global existence of a solution was demonstrated by Sideris and Thomases in [1, 2] through two different methods. It is more challenging to obtain the global existence for the two-dimensional incompressible elastodynamics due to the weaker dispersive decay. In [10], the authors proved almost global existence for a two-dimensional incompressible system in Eulerian coordinates. By introducing the concept of strong null condition and observing that the incompressible elastodynamics automatically satisfies such strong null structure in Lagrangian coordinates, Lei [11] successfully proved the global well-posedness for two-dimensional incompressible elastodynamics by the method of Klainerman and Alinhac's ghost weight method [12]. We also see [13] for a different approach using the spacetime resonance method. All the aforementioned works considered the homogeneous fluids. In [14], the authors established the global well-posedness for the three-dimensional inhomogeneous incompressible elastodynamics in Lagrangian coordinates. It is noteworthy that the upper bound of the highest-order generalized energy in those studies depends on time. Utilizing the Klainerman's generalized energy method, an analysis of the inherent structure of the system and the ghost weight method, [15, 16] established the uniform bound for the highest-order generalized energy estimates for two-dimensional and three-dimensional incompressible elastodynamics, respectively. Based on the above foundational works, it is natural to verify the uniform bound for the highest-order generalized energy for three-dimensional inhomogeneous incompressible isotropic elastodynamics. To establish the time growth of the Sobolev norm of classical solution, two novel methods are presented in this paper. First, based on the Sobolev embedding inequality and the structure of the system, the refined decay rates were derived for the solution in the domain away from the light cone. Second, we apply the KSS-type estimate to overcome the difficulties posed by insufficient time decay resulting from density perturbation.

This paper is organized as follows. In Section 2, we introduce the system of three-dimensional inhomogeneous isotropic elastodynamics and define the notations utilized throughout this paper. Besides, the main result along with several useful lemmas are presented in this section. The energy estimates are discussed in Section 3.

2. Preliminaries

We first formulate the inhomogeneous isotropic elastodynamics and denote some notations that are used frequently in this paper.

2.1. Equations and notations

For any given smooth flow map $X(t, x)$, we call it incompressible if

$$\int_{\Omega} dx = \int_{\Omega_t} dX, \quad \Omega_t = \{X(t, x) \mid x \in \Omega\}$$

for any smooth bounded connected domain Ω , which yields that

$$\det(\nabla X) = 1.$$

Denote

$$X(t, x) = x + \mathbf{v}(t, x).$$

Simple calculation shows that the incompressible condition is equivalent to

$$\nabla \cdot \mathbf{v} + \frac{1}{2}[(\text{tr} \nabla \mathbf{v})^2 - \text{tr}(\nabla \mathbf{v})^2] + \det(\nabla \mathbf{v}) = 0. \quad (2.1)$$

Without loss of generality, we assume that the density of fluid is a small perturbation around the constant state 1, that is, $\rho(x) = 1 + \eta(x)$. For the inhomogeneous isotropic material, the motion of the elastic fluid in the Lagrangian coordinate is determined by

$$\mathcal{L}(X; T, \Omega) = \int_0^T \int_{\Omega} \left(\frac{1}{2} \rho(x) |\partial_t X|^2 - W(\nabla X) + p(t, x) [\det(\nabla X) - 1] \right) dx dt. \quad (2.2)$$

Here $W(\nabla X) \in C^\infty$ is the strain energy function. $p(t, x)$ is a Lagrangian multiplier that is used to force the flow maps to be incompressible. To simplify the presentation, we only study the typical Hookean case for which the strain energy functional is given by

$$W(\nabla X) = \frac{1}{2} |\nabla X|^2.$$

By calculating the variation of (2.2), we obtain the equation

$$\rho \partial_t^2 \mathbf{v} - \Delta \mathbf{v} = -(\nabla X)^{-T} \nabla p. \quad (2.3)$$

Now, we introduce the following derivative vector fields

$$\partial_t = \partial_0, \quad \nabla = (\partial_1, \partial_2, \partial_3) \quad \text{and} \quad \partial = (\partial_0, \partial_1, \partial_2, \partial_3).$$

The scaling operator is denoted by

$$S = t\partial_t + r\partial_r.$$

Here, the radial derivative is defined by $\partial_r = \frac{x}{r} \cdot \nabla$, $r = |x|$. The angular momentum operators are denoted by

$$\Omega = x \wedge \nabla.$$

In the application, we usually use the modified rotational operators and scaling operator; that is, for any vector $\mathbf{v}$ and scalar p and ρ , we set

$$\begin{aligned} \widetilde{S}p &= Sp, & \widetilde{S}\rho &= S\rho, & \widetilde{S}\mathbf{v} &= (S - 1)\mathbf{v}, \\ \widetilde{\Omega}_i p &= \Omega_i p, & \widetilde{\Omega}_i \rho &= \Omega_i \rho, & \widetilde{\Omega}_i \mathbf{v} &= \Omega_i \mathbf{v} + U_i \mathbf{v}, \quad i = 1, 2, 3, \end{aligned}$$

where

$$\begin{aligned} U_1 &= e_2 \otimes e_3 - e_3 \otimes e_2, \\ U_2 &= e_3 \otimes e_1 - e_1 \otimes e_3, \\ U_3 &= e_1 \otimes e_2 - e_2 \otimes e_1. \end{aligned}$$

Let

$$\Gamma = (\Gamma_1, \dots, \Gamma_8) = (\partial, \tilde{\Omega}, \tilde{S})$$

and for any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_8) \in \mathbb{N}^8$, we denote

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \dots \Gamma_8^{\alpha_8}.$$

We apply the derivatives Γ^α to the equations (2.1) and (2.3), and then the three-dimensional inhomogeneous isotropic elastodynamics can be written as

$$\begin{aligned} \rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v} = & -(\nabla X)^{-T} \nabla \Gamma^\alpha p - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta (\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T \\ & \cdot (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}) =: N^\alpha \end{aligned} \quad (2.4)$$

with the incompressible condition

$$\nabla \cdot \Gamma^\alpha \mathbf{v} + \sum_{\substack{\beta+\gamma=\alpha \\ i,j=1,2,3,i < j}} C_\alpha^\beta \left(\partial_i \Gamma^\beta \mathbf{v}_i \partial_j \Gamma^\gamma \mathbf{v}_j - \partial_i \Gamma^\beta \mathbf{v}_j \partial_j \Gamma^\gamma \mathbf{v}_i \right) + \sum_{\beta+\gamma+\iota=\alpha} C_\alpha^\beta C_{\alpha-\beta}^\gamma \begin{vmatrix} \partial_1 \Gamma^\beta \mathbf{v}_1 & \partial_2 \Gamma^\beta \mathbf{v}_1 & \partial_3 \Gamma^\beta \mathbf{v}_1 \\ \partial_1 \Gamma^\beta \mathbf{v}_2 & \partial_2 \Gamma^\beta \mathbf{v}_2 & \partial_3 \Gamma^\beta \mathbf{v}_2 \\ \partial_1 \Gamma^\beta \mathbf{v}_3 & \partial_2 \Gamma^\beta \mathbf{v}_3 & \partial_3 \Gamma^\beta \mathbf{v}_3 \end{vmatrix} = 0. \quad (2.5)$$

Here the binomial coefficient C_α^β is given by

$$C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

2.2. Main result

We denote the Klainerman's generalized energy by

$$\mathcal{E}_\kappa(t) = \sum_{|\alpha| \leq \kappa-1} \int_{\mathbb{R}^3} (|\partial_t \Gamma^\alpha \mathbf{v}|^2 + |\nabla \Gamma^\alpha \mathbf{v}|^2) dx.$$

We also define the weighted energy norm

$$\mathcal{X}_\kappa(t) = \sum_{|\alpha| \leq \kappa-2} \int_{\mathbb{R}^3} \langle t-r \rangle^2 |\partial^2 \Gamma^\alpha \mathbf{v}|^2 dx.$$

To describe the space of initial data, we introduce the time-independent analogue of Γ as

$$\Lambda = (\nabla, r \partial_r - 1, \tilde{\Omega})$$

and the space of initial data is defined by

$$H_\Lambda^\kappa = \left\{ (f, g) : \sum_{|\alpha| \leq \kappa-1} \left(\|\Lambda^\alpha f\|_{L^2} + \|\nabla \Lambda^\alpha f\|_{L^2} + \|\Lambda^\alpha g\|_{L^2} \right) < \infty \right\}.$$

As the first step to investigate our problem, we introduce the following lemma, which helps us to solve the additional terms resulting from density perturbation. Let

$$L\mathcal{E}_\kappa(T) = \sum_{|\alpha| \leq \kappa-1} \int_0^T \int_{\mathbb{R}^n} r^{-1+2\mu} \langle r \rangle^{-2\mu'} \left(|\partial \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right)^2 dx dt$$

with $\mu \in (0, \frac{1}{2})$ and $\mu' > \mu$. Without loss of generality, we choose $\mu = \frac{1}{4}$ and $\mu' = \frac{1}{2}$ in this paper. In the case of $\mu' = \mu$, we see $L\mathcal{E}_\kappa(T)$ is the KSS norm, and we denote it by $KSS_\kappa(T)$.

Lemma 1. *Let $f_0 = [r/(1+r)]^{2\mu}$, $f_k = r/(r+2^k)$ with $k \geq 1$, $\mu \in (0, 1/2)$, and $\mathbf{v}$ be the solution to the equation $\partial_t^2 \mathbf{v} - c^2 \Delta \mathbf{v} + h_{ab} \partial_a \partial_b \mathbf{v} = N$ in $[0, T] \times \mathbb{R}^n$ with $h_{ab} = h_{ba}$, $\sum_{0 \leq a, b \leq n} |h_{ab}| \leq \min(1, c^2)/2$ for any integer $n \geq 3$. Then there exists a positive constant C_0 that depends only on the dimension n such that*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} |\partial \mathbf{v}|^2(t) dx + L\mathcal{E}_1(T) + (\ln(2+T))^{-1} KSS_1(T) \\ & \leq C_0 \int_{\mathbb{R}^n} |\partial \mathbf{v}|^2(0) dx + C_0 \int_0^T \int_{\mathbb{R}^n} \left[(|\partial h| + \frac{|h|}{r^{1-2\mu} \langle r \rangle^{2\mu}}) |\partial \mathbf{v}| \left(|\partial \mathbf{v}| + \frac{|\mathbf{v}|}{r} \right) \right] dx dt \\ & \quad + C_0 \left| \int_0^T \int_{\mathbb{R}^n} \partial_t \mathbf{v} \cdot N dx dt \right| + C_0 \sup_{k \geq 0} \left| \int_0^T \int_{\mathbb{R}^n} f_k \left(\partial_r \mathbf{v} + \frac{n-1}{2r} \mathbf{v} \right) \cdot N dx dt \right|, \end{aligned}$$

where $|h| = \sum_{a,b=0}^n |h_{ab}|$ and $|\partial h| = \sum_{a,b,c=0}^n |\partial_c h_{ab}|$.

This lemma can be found in [17]. See also [18, 19] and references therein.

Based on the previous statement, we are ready to show the main result of this paper.

Theorem 1. *Let $W(\nabla X) = \frac{1}{2} |\nabla X|^2$ be an isotropic Hookean strain energy function and $(\mathbf{v}_0, \partial_t \mathbf{v}_0) \in H_\Lambda^\kappa$ with $\kappa \geq 12$. Let $C_0 > 0$ be given constant in Lemma 1. Suppose $\mathbf{v}_0$ satisfies the structural constraint condition (2.1) and*

$$\mathcal{E}_\kappa(0) = \sum_{|\alpha| \leq \kappa-1} (\|\partial_t \Lambda^\alpha \mathbf{v}_0\|_{L^2}^2 + \|\nabla \Lambda^\alpha \mathbf{v}_0\|_{L^2}^2) \leq \varepsilon.$$

If

$$\|\langle r \rangle \Lambda^\alpha \eta\|_{L^2} \leq \delta \quad \text{for } |\alpha| \leq \kappa,$$

then there exist two sufficiently small constants, ε_0, δ_0 and constant C_1 , that depend only on κ and C_0 such that if $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$, the system (2.3) has a unique global classical solution that satisfies

$$\mathcal{E}_\kappa(t) \leq C_1 \varepsilon$$

uniformly for all $t \in [0, +\infty)$.

2.3. Calculus inequalities

In this part, we establish several lemmas that are crucial for the energy estimates. Throughout this paper, we denote $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. The notation $f \lesssim g$ stands for $f \leq Cg$ for some generic constant $C > 0$, which may vary from line to line. In the process of deriving the energy estimates, we usually separate the whole integration domain $\mathbb{R}^3$ into two parts:

$$\mathcal{R} = \{x \in \mathbb{R}^3 : r \leq \langle t \rangle / 8\}, \quad \mathcal{R}^c = \{x \in \mathbb{R}^3 : r > \langle t \rangle / 8\}.$$

We first recall the Sobolev-type inequalities, which were justified in [8].

Lemma 2. For any $\mathbf{v} \in C_0^\infty(\mathbb{R}^3)^3$, $r = |x|$ and $\tilde{r} = |y|$, we have

$$\begin{aligned} \|\langle r \rangle^{\frac{1}{2}} \mathbf{v}(x)\|_{L^\infty} &\lesssim \sum_{|\alpha| \leq 1} \|\nabla \tilde{\Omega}^\alpha \mathbf{v}\|_{L^2}, \\ \|\langle r \rangle \mathbf{v}(x)\|_{L^\infty} &\lesssim \sum_{|\alpha| \leq 1} \|\partial_r \tilde{\Omega}^\alpha \mathbf{v}\|_{L^2(|y| \geq r)}^{\frac{1}{2}} \|\tilde{\Omega}^\alpha \mathbf{v}\|_{L^2(|y| \geq r)}^{\frac{1}{2}}, \\ \|\langle r \rangle \langle t - r \rangle \mathbf{v}(x)\|_{L^\infty} &\lesssim \sum_{|\alpha| \leq 1} \|\langle t - \tilde{r} \rangle \partial_{\tilde{r}} \tilde{\Omega}^\alpha \mathbf{v}\|_{L^2(|y| \geq r)} + \sum_{|\alpha| \leq 2} \|\langle t - \tilde{r} \rangle \tilde{\Omega}^\alpha \mathbf{v}\|_{L^2(|y| \geq r)}. \end{aligned}$$

The following lemma concerns the dispersive decay of solutions in the domain away from the light cone.

Lemma 3. For any $\mathbf{v} \in H^2(\mathbb{R}^3)$, there holds

$$\langle t \rangle \|\mathbf{v}\|_{L^\infty(r \leq \langle t \rangle / 8)} \lesssim \|\mathbf{v}\|_{L^2(r \leq \langle t \rangle / 4)} + \|\langle t - r \rangle \nabla \mathbf{v}\|_{L^2(r \leq \langle t \rangle / 4)} + \|\langle t - r \rangle \nabla^2 \mathbf{v}\|_{L^2(r \leq \langle t \rangle / 4)}.$$

Proof. The proof can be found in Lemma 4.3 in [16]. $\square$

Let $s = \frac{8r}{\langle t \rangle}$. We introduce a radial cutoff function $\xi(s) \in C_0^\infty$ that satisfies

$$\xi(s) = \begin{cases} 1, & s \leq 1, \\ 0, & s \geq 2. \end{cases}$$

It is easy to observe from Lemmas 2 and 3 that

$$\begin{aligned} \langle t \rangle \|\partial \mathbf{v}\|_{L^\infty} &\lesssim \langle t \rangle \|\partial \mathbf{v}\|_{L^\infty(\mathcal{R})} + \|\langle r \rangle (1 - \xi(s)) \partial \mathbf{v}\|_{L^\infty} \\ &\lesssim \|\partial \mathbf{v}\|_{L^2(r \leq \langle t \rangle / 4)} + \|\langle t - r \rangle \nabla \partial \mathbf{v}\|_{L^2(r \leq \langle t \rangle / 4)} + \|\langle t - r \rangle \nabla^2 \partial \mathbf{v}\|_{L^2(r \leq \langle t \rangle / 4)} \\ &\quad + \sum_{\substack{\iota_1, \iota_2 \leq 1}} \|\partial_r^{\iota_1} \tilde{\Omega}^{\iota_2} [(1 - \xi(s)) \partial \mathbf{v}]\|_{L^2} \\ &\lesssim \mathcal{E}_3^{\frac{1}{2}}(t) + \mathcal{X}_3^{\frac{1}{2}}(t). \end{aligned} \tag{2.6}$$

To control the weighted energy norm by the generalized energy norm, the pressure must be estimated via the system (2.4)-(2.5). A similar proof can be found in [14]. For a self-contained presentation, we include its proof below.

Lemma 4. For any integer $\kappa \geq 6$ and multi-index α satisfying $|\alpha| \leq \kappa - 2$, if $\mathcal{E}_{[\kappa/2]+3}(t)$ and $\|\langle r \rangle \Lambda^\alpha \eta\|_{L^2}$ are small. Then there holds

$$\langle t \rangle \|\nabla \Gamma^\alpha p\|_{L^2} + \langle t \rangle \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} \lesssim \delta \mathcal{X}_\kappa^{\frac{1}{2}}(t) + \mathcal{E}_\kappa^{\frac{1}{2}}(t) \mathcal{X}_{[\kappa/2]+3}^{\frac{1}{2}}(t) + \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t) \mathcal{X}_\kappa^{\frac{1}{2}}(t).$$

Proof. It observes from (2.4) that

$$\begin{aligned} \nabla \Gamma^\alpha p &= -(\nabla X)^T (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}) - \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta (\nabla X)^T \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} \\ &\quad - \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}) + \sum_{\iota_1 + \iota_2 = \beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}. \end{aligned} \tag{2.7}$$

We apply $\nabla \Delta^{-1} \nabla \cdot$ to the above equality and take the L^2 norm. By the L^2 boundness of the Riesz operator, one has

$$\begin{aligned} \|\nabla \Gamma^\alpha p\|_{L^2} &\lesssim \|\nabla \Delta^{-1} \nabla \cdot (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} + \|(\nabla \mathbf{v})^T (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|(\nabla X)^T \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} \\ &+ \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|(\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^2}. \end{aligned} \quad (2.8)$$

Special attention is paid to the first term on the right-hand side of (2.8). We apply the derivative operator $(\partial_t^2 - \Delta)$ to the incompressible condition (2.5). Without loss of generality, we assume at least one order derivative operator of $(\partial_t^2 - \Delta)$ works on the first component of the velocity field in the second line of (2.5). Then we obtain

$$\begin{aligned} &(\partial_t^2 - \Delta)(\nabla \cdot \Gamma^\alpha \mathbf{v}) \\ &= - \sum_{\substack{\beta+\gamma=\alpha \\ i,j=1,2,3,i < j}} C_\alpha^\beta (\partial_t^2 - \Delta) (\partial_i \Gamma^\beta \mathbf{v}_i \partial_j \Gamma^\gamma \mathbf{v}_j - \partial_i \Gamma^\beta \mathbf{v}_j \partial_j \Gamma^\gamma \mathbf{v}_i) - \sum_{\beta+\gamma+\iota=\alpha} C_\alpha^\beta C_{\alpha-\beta}^\gamma (\partial_t^2 - \Delta) \begin{vmatrix} \partial_1 \Gamma^\beta \mathbf{v}_1 & \partial_2 \Gamma^\beta \mathbf{v}_1 & \partial_3 \Gamma^\beta \mathbf{v}_1 \\ \partial_1 \Gamma^\beta \mathbf{v}_2 & \partial_2 \Gamma^\beta \mathbf{v}_2 & \partial_3 \Gamma^\beta \mathbf{v}_2 \\ \partial_1 \Gamma^\beta \mathbf{v}_3 & \partial_2 \Gamma^\beta \mathbf{v}_3 & \partial_3 \Gamma^\beta \mathbf{v}_3 \end{vmatrix} \\ &= \sum_{\substack{\beta+\gamma=\alpha \\ m+n=1, i,j=0,1,2,3}} C_\alpha^\beta \nabla \cdot \begin{pmatrix} \partial_i^{m+1} \Gamma^\beta \mathbf{v}_1 \partial_2 \partial_i^n \Gamma^\gamma \mathbf{v}_2 + \partial_i^{m+1} \Gamma^\beta \mathbf{v}_1 \partial_3 \partial_i^n \Gamma^\gamma \mathbf{v}_3 - \partial_2 \partial_i^m \Gamma^\beta \mathbf{v}_1 \partial_i^{n+1} \Gamma^\gamma \mathbf{v}_2 - \partial_3 \partial_i^m \Gamma^\beta \mathbf{v}_1 \partial_i^{n+1} \Gamma^\gamma \mathbf{v}_3 \\ -\partial_i^{m+1} \Gamma^\beta \mathbf{v}_1 \partial_1 \partial_i^n \Gamma^\gamma \mathbf{v}_2 + \partial_i^{m+1} \Gamma^\beta \mathbf{v}_2 \partial_3 \partial_i^n \Gamma^\gamma \mathbf{v}_3 + \partial_1 \partial_i^m \Gamma^\beta \mathbf{v}_1 \partial_i^{n+1} \Gamma^\gamma \mathbf{v}_2 - \partial_3 \partial_i^m \Gamma^\beta \mathbf{v}_2 \partial_i^{n+1} \Gamma^\gamma \mathbf{v}_3 \\ -\partial_i^{m+1} \Gamma^\beta \mathbf{v}_2 \partial_2 \partial_i^n \Gamma^\gamma \mathbf{v}_3 - \partial_i^{m+1} \Gamma^\beta \mathbf{v}_1 \partial_1 \partial_i^n \Gamma^\gamma \mathbf{v}_3 + \partial_2 \partial_i^m \Gamma^\beta \mathbf{v}_2 \partial_i^{n+1} \Gamma^\gamma \mathbf{v}_3 + \partial_1 \partial_i^m \Gamma^\beta \mathbf{v}_1 \partial_i^{n+1} \Gamma^\gamma \mathbf{v}_3 \end{pmatrix} \\ &+ \sum_{\substack{\beta+\gamma+\iota=\alpha \\ m_2+m_3=1, i,j=0,1,2,3}} C_\alpha^\beta C_{\alpha-\beta}^\gamma \nabla \cdot \left[\partial_i^{m_1+1} \Gamma^\beta \mathbf{v}_1 \begin{pmatrix} \partial_2 \partial_i^{m_2} \Gamma^\gamma \mathbf{v}_2 \partial_3 \partial_i^{m_3} \Gamma^\iota \mathbf{v}_3 - \partial_3 \partial_i^{m_2} \Gamma^\gamma \mathbf{v}_2 \partial_2 \partial_i^{m_3} \Gamma^\iota \mathbf{v}_3 \\ \partial_3 \partial_i^{m_2} \Gamma^\gamma \mathbf{v}_2 \partial_1 \partial_i^{m_3} \Gamma^\iota \mathbf{v}_3 - \partial_1 \partial_i^{m_2} \Gamma^\gamma \mathbf{v}_2 \partial_3 \partial_i^{m_3} \Gamma^\iota \mathbf{v}_3 \\ \partial_1 \partial_i^{m_2} \Gamma^\gamma \mathbf{v}_2 \partial_2 \partial_i^{m_3} \Gamma^\iota \mathbf{v}_3 - \partial_2 \partial_i^{m_2} \Gamma^\gamma \mathbf{v}_2 \partial_1 \partial_i^{m_3} \Gamma^\iota \mathbf{v}_3 \end{pmatrix} \right]. \end{aligned}$$

Based on the above equality, we handle the first term on the right-hand side of (2.8) as follows

$$\begin{aligned} &\|\nabla \Delta^{-1} \nabla \cdot (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} \\ &\lesssim \|\nabla \Delta^{-1} [(\rho \partial_t^2 - \Delta)(\nabla \cdot \Gamma^\alpha \mathbf{v})]\|_{L^2} + \sum_{i=1,2,3} \|\nabla \Delta^{-1} (\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i)\|_{L^2} \\ &\lesssim (1 + \|\nabla \mathbf{v}\|_{L^\infty}) \|\nabla \mathbf{v}\|_{L^\infty} \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i,j=1,2,3,i \neq j}} \|\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j\|_{L^2} \\ &+ \sum_{\substack{\beta+\gamma+\iota=\alpha, |\beta| \neq |\alpha| \\ i,j,k=1,2,3,i \neq j \neq k}} \|\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \partial \Gamma^\iota \mathbf{v}_k\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i,j=1,2,3,i \neq j}} \|\nabla \Delta^{-1} (\partial \eta \partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j)\|_{L^2} \\ &+ \sum_{\substack{\beta+\gamma+\iota=\alpha, |\beta| \neq |\alpha| \\ i,j,k=1,2,3,i \neq j \neq k}} \|\nabla \Delta^{-1} (\partial \eta \partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \partial \Gamma^\iota \mathbf{v}_k)\|_{L^2} + \sum_{i=1,2,3} \|\nabla \Delta^{-1} (\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i)\|_{L^2} \\ &+ \sum_{\substack{i,j=1,2,3 \\ i \neq j}} \|\nabla \Delta^{-1} (\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i \partial \mathbf{v}_j)\|_{L^2} + \sum_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k}} \|\nabla \Delta^{-1} (\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i \partial \mathbf{v}_j \partial \mathbf{v}_k)\|_{L^2}. \end{aligned} \quad (2.9)$$

By (2.8), (2.9), the Sobolev embedding inequality, and the smallness of $\|\nabla \eta\|_{L^3}$, we have

$$\|\nabla \Gamma^\alpha p\|_{L^2} \lesssim (1 + \|\nabla \mathbf{v}\|_{L^\infty}) \|\nabla \mathbf{v}\|_{L^\infty} \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i,j=1,2,3,i \neq j}} \|\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j\|_{L^2}$$

$$\begin{aligned}
& + \sum_{\substack{\beta+\gamma+\iota=\alpha, |\beta|\neq|\alpha| \\ 1, j, k=1, 2, 3, i\neq j\neq k}} \|\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \partial \Gamma^\iota \mathbf{v}_k\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} \|\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} \|(\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^2} + \sum_{i=1, 2, 3} \|\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i\|_{L^{\frac{6}{5}}} \\
& + \sum_{\substack{i, j=1, 2, 3 \\ i\neq j}} \|\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i \partial \mathbf{v}_j\|_{L^{\frac{6}{5}}} + \sum_{\substack{i, j, k=1, 2, 3 \\ i\neq j}} \|\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i \partial \mathbf{v}_j \partial \mathbf{v}_k\|_{L^{\frac{6}{5}}}.
\end{aligned}$$

By (2.4), the above inequality and the smallness of $\mathcal{E}_{[\kappa/2]+3}(t)$, one obtains

$$\begin{aligned}
& \|\nabla \Gamma^\alpha p\|_{L^2} + \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} \\
& \lesssim \sum_{\substack{\beta+\gamma+\iota=\alpha, |\beta|\neq|\alpha| \\ i, j, k=1, 2, 3, i\neq j}} \|\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \partial \Gamma^\iota \mathbf{v}_k\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} \|\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} \|(\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^2} + \sum_{i=1, 2, 3} \|\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i\|_{L^{\frac{6}{5}}} \\
& + \sum_{\substack{i, j=1, 2, 3 \\ i\neq j}} \|\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i \partial \mathbf{v}_j\|_{L^{\frac{6}{5}}} + \sum_{\substack{i, j, k=1, 2, 3 \\ i\neq j\neq k}} \|\partial \eta \partial_t^2 \Gamma^\alpha \mathbf{v}_i \partial \mathbf{v}_j \partial \mathbf{v}_k\|_{L^{\frac{6}{5}}}.
\end{aligned}$$

We deduce from Lemma 2 that

$$\begin{aligned}
& \langle t \rangle \|\nabla \Gamma^\alpha p\|_{L^2} + \langle t \rangle \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \|\langle r \rangle \partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\langle r \rangle \langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} \\
& + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota|<|\beta|<|\alpha|}} \|\langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \|\langle r \rangle \partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\iota \mathbf{v}\|_{L^\infty} + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\iota|\leq|\gamma|}} \|\langle r \rangle \langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} \|\partial \Gamma^\iota \mathbf{v}\|_{L^\infty} \\
& + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\gamma|\leq|\iota|}} \|\langle r \rangle \langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\iota \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\langle r \rangle \Gamma^\gamma \eta\|_{L^\infty} \|\langle t-r \rangle \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\Gamma^\gamma \eta\|_{L^2} \|\langle r \rangle \langle t-r \rangle \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} + \|\langle r \rangle \nabla \eta\|_{L^3} (1 + \|\nabla \mathbf{v}\|_{L^\infty})^2 \|\langle t-r \rangle \partial_t^2 \Gamma^\alpha \mathbf{v}\|_{L^2} \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\langle r \rangle \nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\langle t-r \rangle (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^2} \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^2} \|\langle r \rangle \langle t-r \rangle (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^\infty} \\
& \lesssim \delta \mathcal{X}_\kappa^{\frac{1}{2}}(t) + \mathcal{E}_\kappa^{\frac{1}{2}}(t) \mathcal{X}_{[\kappa/2]+3}^{\frac{1}{2}}(t) + \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t) \mathcal{X}_\kappa^{\frac{1}{2}}(t).
\end{aligned}$$

It completes the proof. $\square$

As an application of the above result, we establish the estimate of weighted energy.

Lemma 5. *Let $\mathbf{v} \in H_\Gamma^\kappa(\mathbb{R}^3)$ be the solution to the system (2.3) with the constraint condition (2.1). For any integer $\kappa \geq 6$, if $\mathcal{E}_\kappa(t)$ and $\|\langle r \rangle \Lambda^\alpha \eta\|_{L^2}$ are small, then we have*

$$\mathcal{X}_\kappa(t) \lesssim \mathcal{E}_\kappa(t).$$

Proof. For any multi-index α satisfying $|\alpha| \leq \kappa - 2$, we apply Γ^α to Lemma 3.3 in [11] to get

$$\mathcal{X}_\kappa(t) \lesssim \mathcal{E}_\kappa(t) + t \|\partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2}^2.$$

It follows from the above inequality and Lemma 4 that

$$\mathcal{X}_\kappa(t) \lesssim \mathcal{E}_\kappa(t) + \delta^2 \mathcal{X}_\kappa(t) + \mathcal{E}_{[\kappa/2]+3}(t) \mathcal{X}_\kappa(t) + \mathcal{E}_\kappa(t) \mathcal{X}_{[\kappa/2]+3}(t).$$

By the smallness of δ and $\mathcal{E}_\kappa(t)$, we arrive at the lemma. $\square$

In preparation for the energy estimates, more detailed analysis of pressure is needed. In what follows, we always assume that $\mathcal{E}_\kappa(t)$ and $\|\langle r \rangle \Lambda^\alpha \eta\|_{L^2}$ are small.

Lemma 6. *For any integer $\kappa \geq 8$ and multi-index α satisfying $|\alpha| \leq \kappa - 1$, we have*

$$\|\nabla \Gamma^\alpha p\|_{L^2} \lesssim \mathcal{E}_\kappa^{\frac{1}{2}}(t) \quad (2.10)$$

and

$$\langle t \rangle \|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R}^c)} + \langle t \rangle \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R}^c)} \lesssim \mathcal{E}_\kappa^{\frac{1}{2}}(t). \quad (2.11)$$

Proof. Following the calculations in Lemma 4, we arrive at (2.9). Special attention is paid to the last three terms on the right-hand side of (2.9). From (2.4), one has

$$\begin{aligned} \partial_t^2 \Gamma^\alpha \mathbf{v} &= \rho^{-1} \Delta \Gamma^\alpha \mathbf{v} - \rho^{-1} (\nabla X)^{-T} \nabla \Gamma^\alpha p - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \rho^{-1} \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} \\ &\quad - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \rho^{-1} (\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}). \end{aligned}$$

We use the above equality and the Sobolev embedding inequality to solve the last three terms on the right-hand side of (2.9) by

$$\begin{aligned} &\|\nabla \Delta^{-1} [\rho^{-1} \partial \eta (1 + \partial \mathbf{v})^2 \Delta \Gamma^\alpha \mathbf{v}]\|_{L^2} + \|\nabla \Delta^{-1} [\rho^{-1} \partial \eta (1 + \partial \mathbf{v})^2 (\nabla X)^{-T} \nabla \Gamma^\alpha p]\|_{L^2} \\ &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|\nabla \Delta^{-1} [\rho^{-1} \partial \eta (1 + \partial \mathbf{v})^2 (\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})]\|_{L^2} \\ &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|\nabla \Delta^{-1} [\rho^{-1} \partial \eta (1 + \partial \mathbf{v})^2 \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}]\|_{L^2} \\ &\lesssim \|\partial \eta (1 + \partial \mathbf{v})^2 \nabla \Gamma^\alpha \mathbf{v}\|_{L^2} + \sum_{i+j+k=1} \|\partial^i (\rho^{-1}) \partial^j \partial \eta \partial^k (1 + \nabla \mathbf{v})^2 \nabla \Gamma^\alpha \mathbf{v}\|_{L^{\frac{6}{5}}} + \|\partial \eta (1 + \partial \mathbf{v})^2 (\nabla X)^{-T} \nabla \Gamma^\alpha p\|_{L^{\frac{6}{5}}} \\ &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|\partial \eta (1 + \partial \mathbf{v})^2 (\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^{\frac{6}{5}}} \\ &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|\partial \eta (1 + \partial \mathbf{v})^2 \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^{\frac{6}{5}}}. \end{aligned}$$

Utilizing (2.4), (2.8), the above estimate, and the Sobolev embedding inequality, we arrive at

$$\begin{aligned}
& \|\nabla \Gamma^\alpha p\|_{L^2} + \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|,|\iota|<|\beta|<|\alpha|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\iota \mathbf{v}\|_{L^\infty} \\
& \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\iota|\leq|\gamma|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} \|\partial \Gamma^\iota \mathbf{v}\|_{L^\infty} + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\gamma|\leq|\iota|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\iota \mathbf{v}\|_{L^2} \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\Gamma^\gamma \eta\|_{L^\infty} \|\partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\Gamma^\gamma \eta\|_{L^2} \|\partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} + \sum_{1\leq|\iota|\leq 3} \|\nabla^\iota \eta\|_{L^2} \|\nabla \Gamma^\alpha \mathbf{v}\|_{L^2} \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}\| + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^2} \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^2} \|\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}\| + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^\infty} \\
& \lesssim \delta \mathcal{E}_\kappa^{\frac{1}{2}}(t) + \mathcal{E}_\kappa^{\frac{1}{2}}(t) \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t). \tag{2.12}
\end{aligned}$$

The smallness of δ and $\mathcal{E}_\kappa(t)$ leads to (2.10).

To verify (2.11), we use (2.12), Lemma 2, and the smallness of $\mathcal{E}_\kappa(t)$ to get

$$\begin{aligned}
& \langle t \rangle \|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R}^c)} + \langle t \rangle \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R}^c)} \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \|\langle r \rangle \partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\langle r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\iota|\leq|\gamma|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} \|\langle r \rangle \partial \Gamma^\iota \mathbf{v}\|_{L^\infty} \\
& \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\iota|\leq|\gamma|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\langle r \rangle \partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\iota \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|,|\iota|<|\beta|<|\alpha|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \|\langle r \rangle \partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\iota \mathbf{v}\|_{L^\infty} \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\langle r \rangle \Gamma^\gamma \eta\|_{L^\infty} \|\partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\Gamma^\gamma \eta\|_{L^2} \|\langle r \rangle \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} + \sum_{1\leq|\iota|\leq 3} \|\langle r \rangle \nabla^\iota \eta\|_{L^2} \|\nabla \Gamma^\alpha \mathbf{v}\|_{L^2} \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \|\langle r \rangle \nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}\| + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^2} \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\leq|\gamma|}} \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^2} \|\langle r \rangle (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}) + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^\infty} \\
& \lesssim \mathcal{E}_\kappa^{\frac{1}{2}}(t) \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t) + \delta \mathcal{E}_\kappa^{\frac{1}{2}}(t).
\end{aligned}$$

The smallness of δ and $\mathcal{E}_\kappa(t)$ implies (2.11). $\square$

In the subsequent part, we present the improved decay properties for the third-order spatial derivatives of unknown variables in the domain away from the light cone.

Lemma 7. *For any integer $\kappa \geq 10$ and multi-index α satisfying $|\alpha| \leq [\kappa/2]$, it holds that*

$$\langle t \rangle^2 \|\nabla^3 \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} \lesssim \mathcal{E}_{[\kappa/2]+5}^{\frac{1}{2}}(t).$$

Proof. We apply the derivative operator ∇ to the equation (2.4) to get

$$\begin{aligned} \nabla \Delta \Gamma^\alpha \mathbf{v} = & \nabla \eta \partial_t^2 \Gamma^\alpha \mathbf{v} + \rho \partial_t^2 \nabla \Gamma^\alpha \mathbf{v} + \nabla (\nabla X)^{-T} \nabla \Gamma^\alpha p + (\nabla X)^{-T} \nabla \nabla \Gamma^\alpha p + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \nabla \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} \\ & + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \Gamma^\gamma \eta \partial_t^2 \nabla \Gamma^\beta \mathbf{v} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \nabla [(\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})]. \end{aligned}$$

By multiplying the above equality by $t^2 \xi(s)$ and taking the L^2 inner product, one has

$$\begin{aligned} & \int_{\mathbb{R}^3} |t^2 \xi(s) \nabla \Delta \Gamma^\alpha \mathbf{v}|^2 dx \\ & \leq 7 \int_{\mathbb{R}^3} |t^2 \xi(s) \nabla \eta \partial_t^2 \Gamma^\alpha \mathbf{v}|^2 dx + 7 \int_{\mathbb{R}^3} |t^2 \xi(s) \rho \partial_t^2 \nabla \Gamma^\alpha \mathbf{v}|^2 dx + 7 \int_{\mathbb{R}^3} |t^2 \xi(s) \nabla (\nabla X)^{-T} \nabla \Gamma^\alpha p|^2 dx \\ & \quad + 7 \int_{\mathbb{R}^3} |t^2 \xi(s) (\nabla X)^{-T} \nabla^2 \Gamma^\alpha p|^2 dx + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} 7 \int_{\mathbb{R}^3} |C_\alpha^\beta t^2 \xi(s) \nabla \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}|^2 dx \\ & \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} 7 \int_{\mathbb{R}^3} |C_\alpha^\beta t^2 \xi(s) \Gamma^\gamma \eta \partial_t^2 \nabla \Gamma^\beta \mathbf{v}|^2 dx + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} 7 \int_{\mathbb{R}^3} |C_\alpha^\beta t^2 \xi(s) \nabla [(\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T \\ & \quad \cdot (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})]|^2 dx. \end{aligned} \quad (2.13)$$

Since for any $f \in H^2(\mathbb{R}^3)$, the following Sobolev embedding inequality holds true

$$\|f\|_{L^\infty} \lesssim \|\nabla f\|_{L^2}^{\frac{1}{2}} \|\nabla^2 f\|_{L^2}^{\frac{1}{2}}. \quad (2.14)$$

By Lemma 2, the first term on the right-hand side of (2.13) is estimated by

$$\begin{aligned} & 7 \int_{\mathbb{R}^3} |t^2 \xi(s) \nabla \eta \partial_t^2 \Gamma^\alpha \mathbf{v}|^2 dx \\ & \lesssim \int_{\mathbb{R}^3} |tr \xi(s) \nabla \eta \partial_r \partial_t \Gamma^\alpha \mathbf{v}|^2 dx + \int_{\mathbb{R}^3} |t \xi(s) \nabla \eta \partial_t \tilde{S} \Gamma^\alpha \mathbf{v}|^2 dx \\ & \lesssim \left(\|\xi(s) \langle r \rangle \langle t-r \rangle \partial_r \partial_t \Gamma^\alpha \mathbf{v}\|_{L^\infty}^2 + \|t \xi(s) \partial_t \tilde{S} \Gamma^\alpha \mathbf{v}\|_{L^\infty}^2 \right) \|\nabla \eta\|_{L^2}^2 \\ & \lesssim \left(\|\langle r \rangle \langle t-r \rangle \partial_r \partial_t \Gamma^\alpha \mathbf{v}\|_{L^\infty}^2 + \|\nabla(t \xi(s) \partial_t \tilde{S} \Gamma^\alpha \mathbf{v})\|_{L^2} \|\nabla^2(t \xi(s) \partial_t \tilde{S} \Gamma^\alpha \mathbf{v})\|_{L^2} \right) \|\nabla \eta\|_{L^2}^2 \\ & \lesssim \left[\|\langle r \rangle \langle t-r \rangle \partial_r \partial_t \Gamma^\alpha \mathbf{v}\|_{L^\infty}^2 + \left(\|\xi'(s) \partial_t \tilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2} + \|\xi(s) \langle t-r \rangle \partial_t \nabla \tilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2} \right) \right. \\ & \quad \cdot \left(\langle t \rangle^{-1} \|\xi''(s) \partial_t \tilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2} + \|\xi'(s) \partial_t \nabla \tilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2} + \|\xi(s) \langle t-r \rangle \partial_t \nabla^2 \tilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2} \right) \left. \right] \|\nabla \eta\|_{L^2}^2 \\ & \lesssim \delta^2 \mathcal{E}_{[\kappa/2]+4}(t). \end{aligned} \quad (2.15)$$

By a similar argument, we handle the second term on the right-hand side of (2.13) as follows

$$\begin{aligned} & 7 \int_{\mathbb{R}^3} |t^2 \xi(s) \rho \partial_t^2 \nabla \Gamma^\alpha \mathbf{v}|^2 dx \\ & \leq 7 \int_{\mathbb{R}^3} |tr \xi(s) \rho \partial_r \partial_t \nabla \Gamma^\alpha \mathbf{v}|^2 dx + C \int_{\mathbb{R}^3} \left(|t \xi(s) \rho \partial_t \nabla \tilde{S} \Gamma^\alpha \mathbf{v}|^2 + |t \xi(s) \rho \partial_t \nabla \Gamma^\alpha \mathbf{v}|^2 \right) dx \end{aligned}$$

$$\begin{aligned}
&\leq 7 \int_{\mathbb{R}^3} |r^2 \xi(s) \rho \partial_r^2 \nabla \Gamma^\alpha \mathbf{v}|^2 dx + C \int_{\mathbb{R}^3} (|t \xi(s) \rho \partial_t \nabla \tilde{S} \Gamma^\alpha \mathbf{v}|^2 + |t \xi(s) \rho \partial_t \nabla \Gamma^\alpha \mathbf{v}|^2 \\
&\quad + |\xi(s) \rho r \partial_r \nabla \tilde{S} \Gamma^\alpha \mathbf{v}|^2 + |\xi(s) \rho r \partial_r \nabla \Gamma^\alpha \mathbf{v}|^2) dx \\
&\leq 7 \int_{\mathbb{R}^3} |r^2 \xi(s) \rho \nabla^3 \Gamma^\alpha \mathbf{v}|^2 dx + C (\|\langle t-r \rangle \partial \nabla \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})}^2 + \|\langle t-r \rangle \partial \nabla \tilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})}^2) \\
&\leq \frac{7}{16} \int_{\mathbb{R}^3} |t^2 \xi(s) \nabla^3 \Gamma^\alpha \mathbf{v}|^2 dx + C \mathcal{E}_{[\kappa/2]+3}(t).
\end{aligned} \tag{2.16}$$

In view of (2.7), the smallness of δ and $\mathcal{E}_\kappa(t)$, one has

$$\begin{aligned}
&\|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})} + \langle t \rangle \|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})} \\
&\lesssim \|(1 + \langle t \rangle)(\nabla X)^T (\partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2(\mathcal{R})} + \sum_{\beta+\gamma=\alpha} \|(1 + \langle t \rangle)(\nabla X)^T \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})} \\
&\quad + \sum_{\beta+\gamma=\alpha} \|(1 + \langle t \rangle)(\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^2(\mathcal{R})} \\
&\lesssim \mathcal{E}_{[\kappa/2]+2}^{\frac{1}{2}}(t).
\end{aligned}$$

We substitute the above inequality into the third term on the right-hand side of (2.13) to get

$$\begin{aligned}
&7 \int_{\mathbb{R}^3} |t^2 \xi(s) \nabla (\nabla X)^{-T} \nabla \Gamma^\alpha p|^2 dx \\
&\lesssim \langle t \rangle^2 \|\nabla^2 \mathbf{v}\|_{L^\infty(\mathcal{R})}^2 \langle t \rangle^2 \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{R})}^2 \|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})}^2 + \langle t \rangle^2 \|\nabla^2 \mathbf{v}\|_{L^\infty(\mathcal{R})}^2 \langle t \rangle^2 \|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})}^2 \\
&\lesssim \mathcal{E}_{[\kappa/2]+2}(t).
\end{aligned}$$

We come back to the fourth term on the right-hand side of (2.13). We apply the divergence operator to the equality (2.7) and multiply t^2 on both sides of the resulting equality. By taking the L^2 inner product, one has

$$\begin{aligned}
\|t^2 \xi(s) \Delta \Gamma^\alpha p\|_{L^2} &\lesssim \|t^2 \xi(s) \nabla \cdot [(\nabla \mathbf{v})^T (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})]\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|t^2 \xi(s) \nabla \cdot [(\nabla \mathbf{v})^T \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}]\|_{L^2} \\
&\quad + \|t^2 \xi(s) \nabla \cdot (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|t^2 \xi(s) \nabla \cdot (\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v})\|_{L^2} \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|t^2 \xi(s) \nabla \cdot [(\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})]\|_{L^2}.
\end{aligned} \tag{2.17}$$

For the first two terms on the right-hand side of (2.17), we deduce from Lemma 3 that

$$\begin{aligned}
&\|t^2 \xi(s) \nabla \cdot [(\nabla \mathbf{v})^T (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})]\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|t^2 \xi(s) \nabla \cdot [(\nabla \mathbf{v})^T \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}]\|_{L^2} \\
&\lesssim \sum_{i+j=1} \langle t \rangle \|\nabla^i (\nabla \mathbf{v})^T\|_{L^\infty(\mathcal{R})} \|\langle t-r \rangle \nabla^j (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2(\mathcal{R})} \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \leq |\gamma| \\ i+j+k=1}} \langle t \rangle \|\nabla^i (\nabla \mathbf{v})^T\|_{L^\infty(\mathcal{R})} \|\nabla^j \Gamma^\gamma \eta\|_{L^2(\mathcal{R})} \|\partial_t^2 \nabla^k \Gamma^\beta \mathbf{v}\|_{L^\infty(\mathcal{R})}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\beta+\gamma=\alpha, |\gamma|<|\beta|<|\alpha| \\ i+j+k=1}} \langle t \rangle \|\nabla^i (\nabla \mathbf{v})^T\|_{L^\infty(\mathcal{R})} \|\nabla^j \Gamma^\gamma \eta\|_{L^\infty(\mathcal{R})} \|\langle t-r \rangle \partial_t^2 \nabla^k \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})} \\
& \lesssim \mathcal{E}_4^{\frac{1}{2}}(t) \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t) + \delta \mathcal{E}_4^{\frac{1}{2}}(t) \mathcal{E}_{[\kappa/2]+5}^{\frac{1}{2}}(t).
\end{aligned}$$

For the third and fourth terms on the right-hand side of (2.17), we have

$$\begin{aligned}
& \|t^2 \xi(s) \nabla \cdot (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|t^2 \xi(s) \nabla \cdot (\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v})\|_{L^2} \\
& \lesssim \|t^2 \xi(s) \nabla \eta \cdot \partial_t^2 \Gamma^\alpha \mathbf{v}\|_{L^2} + \|t^2 \xi(s) [(\rho \partial_t^2 - \Delta)(\nabla \cdot \Gamma^\alpha \mathbf{v})]\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i+j=1}} \|t^2 \xi(s) \nabla^i \Gamma^\gamma \eta \partial_t^2 (\nabla^j \cdot \Gamma^\beta \mathbf{v})\|_{L^2}. \quad (2.18)
\end{aligned}$$

By the definition of $\widetilde{S}$ and the Sobolev embedding inequality, we solve the first term on the right-hand side of (2.18) by

$$\begin{aligned}
\|t^2 \xi(s) \nabla \eta \cdot \partial_t^2 \Gamma^\alpha \mathbf{v}\|_{L^2} & \lesssim \|\nabla \eta\|_{L^3} \|t \xi(s) \partial_t \widetilde{S} \Gamma^\alpha \mathbf{v}\|_{L^6} + \|tr \xi(s) \nabla \eta \partial_r \partial_t \Gamma^\alpha \mathbf{v}\|_{L^2} \\
& \lesssim \|\nabla \eta\|_{L^3} (\|\partial_t \widetilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} + \|\langle t-r \rangle \partial_t \nabla \widetilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})}) + \|\langle r \rangle \nabla \eta\|_{L^\infty} \|\langle t-r \rangle \partial_r \partial_t \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} \\
& \lesssim \delta \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t).
\end{aligned}$$

In terms of (2.5) and the definition of $\widetilde{S}$, the last two terms on the right-hand side of (2.18) are estimated by

$$\begin{aligned}
& \|t^2 \xi(s) [(\rho \partial_t^2 - \Delta)(\nabla \cdot \Gamma^\alpha \mathbf{v})]\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i+j=1}} \|t^2 \xi(s) \nabla^i \Gamma^\gamma \eta \partial_t^2 (\nabla^j \cdot \Gamma^\beta \mathbf{v})\|_{L^2} \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ i+j=1}} \|\langle t-r \rangle \partial^i \partial \nabla \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})} \langle t \rangle \|\partial^j \nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} \\
& \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ i+j+k=1}} \|\langle t-r \rangle \partial^i \partial \nabla \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})} \langle t \rangle \|\partial^j \nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} \|\partial^k \nabla \Gamma^\iota \mathbf{v}\|_{L^\infty(\mathcal{R})} \\
& \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ j+k=1}} \|\Gamma^\gamma \eta\|_{L^\infty(\mathcal{R})} \|\langle t-r \rangle \partial^i \partial \nabla \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})} \langle t \rangle \|\partial^j \nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} \|\partial^k \nabla \Gamma^\iota \mathbf{v}\|_{L^\infty(\mathcal{R})} \\
& \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ i+j+k=1}} \|\Gamma^\gamma \eta\|_{L^\infty(\mathcal{R})} \|\langle t-r \rangle \partial^i \partial \nabla \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})} \langle t \rangle \|\partial^j \nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} \|\partial^k \nabla \Gamma^\iota \mathbf{v}\|_{L^\infty(\mathcal{R})} \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|\xi(s) \nabla \Gamma^\gamma \eta\|_{L^3} \langle t \rangle \|\xi(s) \partial_t \widetilde{S} \Gamma^\beta \mathbf{v}\|_{L^6} + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|\langle r \rangle \nabla \Gamma^\gamma \eta\|_{L^\infty(\mathcal{R})} \|\langle t-r \rangle \partial_t \partial_r \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})} \\
& \lesssim (1 + \delta) \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t) (\mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t) + \mathcal{E}_{[\kappa/2]+4}(t)) + \delta \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t).
\end{aligned}$$

Along the same line, the last term on the right-hand side of (2.17) is dealt with by

$$\sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \|t^2 \xi(s) \nabla \cdot [(\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}) + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1 \Gamma^{\iota_2}} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}]\|_{L^2}$$

$$\begin{aligned}
&\lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} \|\langle t-r \rangle \nabla^2 \Gamma^\gamma \mathbf{v}\|_{L^2(\mathcal{R})} \|\langle t \rangle \|\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^\infty(\mathcal{R})} \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} \langle t \rangle \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} \|\langle t-r \rangle \nabla (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^2(\mathcal{R})} \\
&\lesssim \mathcal{E}_{[\kappa/2]+4}(t).
\end{aligned}$$

Combining all the estimates, Lemma 4, and the fact

$$\|t^2 \xi(s) \nabla^2 \Gamma^\alpha p\|_{L^2} \lesssim \|t^2 \xi(s) \Delta \Gamma^\alpha p\|_{L^2} + \|t \xi(s) \nabla \Gamma^\alpha p\|_{L^2},$$

the fourth term on the right-hand side of (2.13) is estimated by

$$\begin{aligned}
7 \int_{\mathbb{R}^3} |t^2 \xi(s) (\nabla X)^{-T} \nabla^2 \Gamma^\alpha p|^2 dx &\lesssim \langle t \rangle^2 \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{R})}^2 \langle t \rangle^2 \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{R})}^2 \|\nabla^2 \Gamma^\alpha p\|_{L^2(\mathcal{R})}^2 \\
&\quad + \langle t \rangle^2 \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{R})}^2 \langle t \rangle^2 \|\nabla^2 \Gamma^\alpha p\|_{L^2(\mathcal{R})}^2 + \|t^2 \nabla^2 \Gamma^\alpha p\|_{L^2(\mathcal{R})}^2 \\
&\lesssim \mathcal{E}_{[\kappa/2]+5}(t).
\end{aligned}$$

We employ the similar method as (2.15) and (2.16) to estimate the fifth and sixth terms on the right-hand side of (2.13) by

$$\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} 7 \int_{\mathbb{R}^3} |C_\alpha^\beta t^2 \xi(s) \nabla \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}|^2 dx + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} 7 \int_{\mathbb{R}^3} |C_\alpha^\beta t^2 \xi(s) \Gamma^\gamma \eta \partial_t^2 \nabla \Gamma^\beta \mathbf{v}|^2 dx \\
&\lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} \int_{\mathbb{R}^3} (|t \xi(s) \nabla \Gamma^\gamma \eta \partial_t \widetilde{S} \Gamma^\beta \mathbf{v}|^2 dx + |t \xi(s) r \nabla \Gamma^\gamma \eta \partial_r \partial_t \Gamma^\beta \mathbf{v}|^2 + |t \xi(s) \Gamma^\gamma \eta \partial_t \nabla \widetilde{S} \Gamma^\beta \mathbf{v}|^2 \\
&\quad + |t \xi(s) \Gamma^\gamma \eta \partial_t \nabla \Gamma^\beta \mathbf{v}|^2 + |t \xi(s) \Gamma^\gamma \eta r \partial_r \partial_t \nabla \Gamma^\beta \mathbf{v}|^2) dx \\
&\lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} (\|\nabla \Gamma^\gamma \eta\|_{L^3}^2 \|t \xi(s) \partial_t \widetilde{S} \Gamma^\beta \mathbf{v}\|_{L^6}^2 + \|\langle r \rangle \xi(s) \nabla \Gamma^\gamma \eta\|_{L^\infty}^2 \|\langle t-r \rangle \partial \nabla \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})}^2 \\
&\quad + \|\Gamma^\gamma \eta\|_{L^\infty(\mathcal{R})}^2 \|\langle t-r \rangle \partial \nabla \widetilde{S} \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})}^2 + \|\Gamma^\gamma \eta\|_{L^\infty(\mathcal{R})}^2 \|\langle t-r \rangle \partial_t \nabla \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})}^2 \\
&\quad + \|\langle r \rangle \xi(s) \Gamma^\gamma \eta\|_{L^\infty}^2 \|\langle t-r \rangle \partial \nabla^2 \Gamma^\beta \mathbf{v}\|_{L^2(\mathcal{R})}^2) \\
&\lesssim \delta^2 \mathcal{E}_{[\kappa/2]+3}(t).
\end{aligned}$$

For the last term on the right-hand side of (2.13), we have

$$\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\neq|\alpha|}} 7 \int_{\mathbb{R}^3} |C_\alpha^\beta t^2 \xi(s) \nabla [(\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})]|^2 dx \\
&\lesssim \sum_{\substack{\beta+\gamma=\alpha, |\beta|\neq|\alpha| \\ i+j\leq 1}} \langle t \rangle^2 \|\nabla^i (\nabla \Gamma^\gamma \mathbf{v})^T\|_{L^\infty(\mathcal{R})}^2 \|\langle t-r \rangle \nabla^j (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})\|_{L^2(\mathcal{R})}^2 \\
&\lesssim \mathcal{E}_{[\kappa/2]+4}(t).
\end{aligned}$$

Collecting the above estimates and the fact

$$\int_{\mathbb{R}^3} |t^2 \xi(s) \nabla^3 \Gamma^\alpha \mathbf{v}|^2 dx \lesssim \int_{\mathbb{R}^3} |t^2 \xi(s) \nabla \Delta \Gamma^\alpha \mathbf{v}|^2 dx + \int_{\mathbb{R}^3} |t \xi(s) \nabla^2 \Gamma^\alpha \mathbf{v}|^2 dx,$$

we complete the proof. $\square$

By Lemma 7, we obtain the following estimate.

Lemma 8. *For any integer $\kappa \geq 12$ and multi-index α satisfying $|\alpha| \leq [\kappa/2]$, there holds*

$$\langle t \rangle^{\frac{3}{2}} \|\partial \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} \lesssim \mathcal{E}_{[\kappa/2]+5}^{\frac{1}{2}}(t), \quad (2.19)$$

$$\langle t \rangle^2 \|\partial^2 \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} \lesssim \mathcal{E}_{[\kappa/2]+6}^{\frac{1}{2}}(t). \quad (2.20)$$

Proof. By the definition of $\widetilde{S}$ and Lemma 7, we have

$$\begin{aligned} \langle t \rangle^2 \|\partial_t \nabla^2 \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} &\lesssim \|\langle t-r \rangle \partial_t \nabla^2 \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} + \|\langle t-r \rangle \nabla^2 \widetilde{S} \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} + \|\langle t-r \rangle \nabla^2 \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} \\ &\quad + \langle t \rangle^2 \|\nabla^3 \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} \\ &\lesssim \mathcal{E}_{[\kappa/2]+5}^{\frac{1}{2}}(t). \end{aligned} \quad (2.21)$$

The inequalities (2.14), (2.21), and Lemma 7 yield that

$$\begin{aligned} \langle t \rangle^{\frac{3}{2}} \|\partial \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} &\lesssim \langle t \rangle^{\frac{3}{2}} \|\nabla(\xi(s) \partial \Gamma^\alpha \mathbf{v})\|_{L^2}^{\frac{1}{2}} \|\nabla^2(\xi(s) \partial \Gamma^\alpha \mathbf{v})\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \left(\|\xi'(s) \partial \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} + \|\xi(s) \langle t-r \rangle \nabla \partial \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} \right) \left(\|\xi''(s) \partial \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} + \|\xi'(s) \langle t-r \rangle \nabla \partial \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} \right. \\ &\quad \left. + \langle t \rangle \|\xi(s) \nabla^2 \partial \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} \right) \\ &\lesssim \mathcal{E}_{[\kappa/2]+5}^{\frac{1}{2}}(t). \end{aligned}$$

To consider (2.20), the definition of $\widetilde{S}$, combined with inequalities (2.14) and (2.21) and Lemmas 3 and 7, implies that

$$\begin{aligned} &\langle t \rangle^2 \|\partial_t^2 \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} + \langle t \rangle^2 \|\partial \nabla \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} \\ &\lesssim \langle t \rangle \|\partial_t^2 \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} + \langle t \rangle \|\partial_t \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} + \langle t \rangle \|\partial_t \widetilde{S} \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} + \langle t \rangle \|r \partial_r \partial_t \Gamma^\alpha \mathbf{v}\|_{L^\infty(\mathcal{R})} \\ &\quad + \langle t \rangle^2 \|\nabla(\xi(s) \partial \nabla \Gamma^\alpha \mathbf{v})\|_{L^2}^{\frac{1}{2}} \|\nabla^2(\xi(s) \partial \nabla \Gamma^\alpha \mathbf{v})\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t) + \sum_{i, |i| \leq 1} \left\| \langle t-r \rangle \partial_r^i \widetilde{\Omega}'(\xi(s) \partial_r \partial_t \Gamma^\alpha \mathbf{v}) \right\|_{L^2} + \left(\|\xi'(s) \langle t-r \rangle \partial \nabla \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} + \|\langle t \rangle^2 \xi(s) \partial \nabla^2 \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} \right) \\ &\quad \cdot \left(\|\xi''(s) \partial \nabla \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} + \|\langle t \rangle \xi'(s) \partial \nabla^2 \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} + \|\langle t \rangle^2 \xi(s) \partial \nabla^3 \Gamma^\alpha \mathbf{v}\|_{L^2}^{\frac{1}{2}} \right) \\ &\lesssim \mathcal{E}_{[\kappa/2]+6}^{\frac{1}{2}}(t), \end{aligned}$$

which implies the desired. $\square$

Before concluding this section, we formulate the following two lemmas, which are utilized in the process of deriving energy estimates.

Lemma 9. For any integer $\kappa \geq 12$ and multi-index α satisfying $|\alpha| \leq [\kappa/2]$, we have

$$\langle t \rangle^2 \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^\infty} \lesssim \mathcal{E}_{[\kappa/2]+5}^{\frac{1}{2}}(t).$$

Proof. We separate two cases to consider this lemma. For the case $x \in \mathcal{R}$, we use the Sobolev embedding inequality (2.14) to get

$$\|\xi(s)(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^\infty} \lesssim \|\nabla[\xi(s)(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})]\|_{L^2}^{\frac{1}{2}} \|\nabla^2[\xi(s)(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})]\|_{L^2}^{\frac{1}{2}}.$$

For the integer i satisfying $1 \leq i \leq 2$, one has

$$\begin{aligned} & \langle t \rangle^2 \|\nabla^i[\xi(s)(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})]\|_{L^2} \\ & \lesssim \sum_{\substack{j+k=i \\ 1 \leq j \leq i}} \langle t \rangle^2 \|\nabla^j \xi(s) \nabla^k(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} + \sum_{1 \leq i \leq 2} \langle t \rangle^2 \|\xi(s) \nabla^i(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2}. \end{aligned} \quad (2.22)$$

We estimate the first term on the right-hand side of (2.22) by

$$\begin{aligned} \sum_{\substack{j+k=i \\ 1 \leq j \leq i}} \langle t \rangle^2 \|\nabla^j \xi(s) \nabla^k(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} & \lesssim \langle t \rangle \|\xi'(s) \nabla(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} + \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} \\ & \lesssim \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t). \end{aligned}$$

By (2.4), we solve the second term on the right-hand side of (2.22) as follows

$$\begin{aligned} & \sum_{1 \leq i \leq 2} \langle t \rangle^2 \|\xi(s) \nabla^i(\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})\|_{L^2} \\ & \lesssim \sum_{\substack{1 \leq i \leq 2 \\ j+k=i, 1 \leq j}} \langle t \rangle^2 \|\xi(s) \nabla^j (\nabla X)^{-T} \nabla^k \nabla \Gamma^\alpha p\|_{L^2} + \sum_{1 \leq i \leq 2} \langle t \rangle^2 \|\xi(s) (\nabla X)^{-T} \nabla^i \nabla \Gamma^\alpha p\|_{L^2} \\ & \quad + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ j+k=i, 1 \leq j \leq i}} \langle t \rangle^2 \|\xi(s) \nabla^j \Gamma^\gamma \eta \partial_t^2 \nabla^k \Gamma^\beta \mathbf{v}\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ 1 \leq i \leq 2}} \langle t \rangle^2 \|\xi(s) \nabla^i [(\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T \\ & \quad \cdot (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})]\|_{L^2}. \end{aligned} \quad (2.23)$$

For the first term on the right-hand side of (2.23), by Lemmas 3 and 4, we have

$$\begin{aligned} \sum_{\substack{1 \leq i \leq 2 \\ j+k=i, 1 \leq j}} \langle t \rangle^2 \|\xi(s) \nabla^j (\nabla X)^{-T} \nabla^k \nabla \Gamma^\alpha p\|_{L^2} & \lesssim \sum_{\substack{1 \leq i \leq 2 \\ j+k=i, 1 \leq j}} \langle t \rangle \|\nabla^j (\nabla X)^{-T}\|_{L^\infty(\mathcal{R})} \langle t \rangle \|\nabla^k \nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})} \\ & \lesssim \mathcal{X}_5^{\frac{1}{2}}(t) (\mathcal{E}_{[\kappa/2]+5}(t) + \delta \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t)). \end{aligned}$$

Adopting the same method as was used in (2.17), we estimate the second term on the right-hand side of (2.23) by

$$\begin{aligned} \sum_{1 \leq i \leq 2} \langle t \rangle^2 \|\xi(s) (\nabla X)^{-T} \nabla^i \nabla \Gamma^\alpha p\|_{L^2} & \lesssim \sum_{1 \leq i \leq 2} \langle t \rangle \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{R})} (1 + \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{R})}) \langle t \rangle \|\nabla^i \nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})} \\ & \quad + \sum_{1 \leq i \leq 2} \langle t \rangle^2 \|\nabla^i \nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})} \end{aligned}$$

$$\lesssim \delta \mathcal{E}_{[\kappa/2]+5}^{\frac{1}{2}}(t) + \mathcal{E}_{[\kappa/2]+5}(t).$$

For the third term on the right-hand side of (2.23), by the definition of $\widetilde{S}$ and (2.6), we obtain

$$\begin{aligned} & \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ j+k=i, 1 \leq i \leq 2}} \langle t \rangle^2 \left\| \xi(s) \nabla^j \Gamma^\gamma \eta \partial_t^2 \nabla^k \Gamma^\beta \mathbf{v} \right\|_{L^2} \\ & \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ j+k=i, 1 \leq i \leq 2}} \langle t \rangle \left(\left\| \xi(s) \nabla^j \Gamma^\gamma \eta \partial_t^2 \nabla^k \Gamma^\beta \mathbf{v} \right\|_{L^2} + \left\| \xi(s) \nabla^j \Gamma^\gamma \eta r \partial_r \partial_t \nabla^k \Gamma^\beta \mathbf{v} \right\|_{L^2} \right) + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ 1 \leq i \leq 2}} \langle t \rangle \left\| \xi(s) \nabla^i \Gamma^\gamma \eta \partial_t \widetilde{S} \Gamma^\beta \mathbf{v} \right\|_{L^2} \\ & \quad + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ j+k=i, 1 \leq k \leq i \leq 2}} \langle t \rangle \left(\left\| \xi(s) \nabla^j \Gamma^\gamma \eta \partial_t \widetilde{S} \nabla^{k-1} \Gamma^\beta \mathbf{v} \right\|_{L^2} + \left\| \xi(s) \nabla^j \Gamma^\gamma \eta \partial_t \nabla^k \Gamma^\beta \mathbf{v} \right\|_{L^2} \right) \\ & \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ j+k=i, 1 \leq i \leq 2}} \left\| \langle r \rangle \xi(s) \nabla^j \Gamma^\gamma \eta \right\|_{L^\infty} \left\| \langle t-r \rangle \partial_t \partial \nabla^k \Gamma^\beta \mathbf{v} \right\|_{L^2(\mathcal{R})} + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ 1 \leq i \leq 2}} \left\| \nabla^i \Gamma^\gamma \eta \right\|_{L^2(\mathcal{R})} \langle t \rangle \left\| \partial_t \widetilde{S} \Gamma^\beta \mathbf{v} \right\|_{L^\infty(\mathcal{R})} \\ & \quad + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ j+k=i, 1 \leq k \leq i \leq 2}} \left\| \nabla^j \Gamma^\gamma \eta \right\|_{L^\infty(\mathcal{R})} \left(\left\| \langle t-r \rangle \partial_t \widetilde{S} \nabla^{k-1} \Gamma^\beta \mathbf{v} \right\|_{L^2(\mathcal{R})} + \left\| \langle t-r \rangle \partial_t \nabla^k \Gamma^\beta \mathbf{v} \right\|_{L^2(\mathcal{R})} \right) \\ & \lesssim \delta \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t). \end{aligned}$$

The last term on the right-hand side of (2.23) is estimated by

$$\begin{aligned} & \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ 1 \leq i \leq 2}} \langle t \rangle^2 \left\| \xi(s) \nabla^i [(\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}) + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1 \iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}] \right\|_{L^2} \\ & \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ 1 \leq i \leq 2, j+k+l=i}} \left\| \nabla^l (\nabla X)^{-T} \right\|_{L^\infty(\mathcal{R})} \langle t \rangle \left\| \nabla^j (\nabla \Gamma^\gamma \mathbf{v})^T \right\|_{L^\infty(\mathcal{R})} \left\| \langle t-r \rangle \nabla^k (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}) + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1 \iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v} \right\|_{L^2(\mathcal{R})} \\ & \lesssim \mathcal{E}_{[\kappa/2]+5}(t). \end{aligned}$$

We verify the case $x \in \mathcal{R}^c$. By Lemmas 2 and 6, we have

$$\begin{aligned} & \langle t \rangle^2 \left\| \rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v} \right\|_{L^\infty(\mathcal{R}^c)} \\ & \lesssim \langle t \rangle \left\| \langle r \rangle (1 - \xi(s)) (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}) \right\|_{L^\infty} \\ & \lesssim \sum_{i, |i| \leq 1} \langle t \rangle \left\| \partial_r^i \widetilde{\Omega}^i [(1 - \xi(s)) (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})] \right\|_{L^2} \\ & \lesssim \sum_{\substack{i+j+k \leq 1 \\ |i_1|+|i_2|+|i_3| \leq 1 \leq j+l_2}} \left\| \partial_r^i \widetilde{\Omega}^{i_1} (1 - \xi(s)) \right\|_{L^\infty(\mathcal{R}^c)} \left\| \langle r \rangle \partial_r^j \widetilde{\Omega}^{i_2} \rho \right\|_{L^\infty(\mathcal{R}^c)} \left\| \partial_r^k \widetilde{\Omega}^{i_3} \partial_t^2 \Gamma^\beta \mathbf{v} \right\|_{L^2(\mathcal{R}^c)} \\ & \quad + \sum_{\substack{i+j \leq 1 \\ |i_1|+|i_2| \leq 1}} \left\| \partial_r^i \widetilde{\Omega}^{i_1} (1 - \xi(s)) \right\|_{L^\infty(\mathcal{R}^c)} \left\| \langle t \rangle (\rho \partial_t^2 \partial_r^j \widetilde{\Omega}^{i_2} \Gamma^\beta \mathbf{v} - \Delta \partial_r^j \widetilde{\Omega}^{i_2} \Gamma^\beta \mathbf{v}) \right\|_{L^2(\mathcal{R}^c)} \\ & \lesssim \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t). \end{aligned}$$

Collecting all of the estimates together, we justify the lemma. $\square$

Lemma 10. For any integer $\kappa \geq 12$ and multi-index α satisfying $|\alpha| \leq [\kappa/2]$, one has

$$\sum_{\beta+\gamma=\alpha} \langle t \rangle^2 \left\| \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} \right\|_{L^\infty} \lesssim \delta \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t).$$

Proof. We deduce from the definition of $\tilde{S}$ that

$$\sum_{\beta+\gamma=\alpha} \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} = \sum_{\beta+\gamma=\alpha} \frac{1}{1+t} \left(\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} + \Gamma^\gamma \eta \partial_t \tilde{S} \Gamma^\beta \mathbf{v} - \Gamma^\gamma \eta r \partial_r \partial_t \Gamma^\beta \mathbf{v} \right).$$

By Lemma 2 and (2.6), one has

$$\begin{aligned} \sum_{\beta+\gamma=\alpha} \langle t \rangle^2 \|\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} &\lesssim \sum_{\beta+\gamma=\alpha} \|\Gamma^\gamma \eta\|_{L^\infty} \|\langle r \rangle \langle t-r \rangle \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} + \sum_{\beta+\gamma=\alpha} \|\Gamma^\gamma \eta\|_{L^\infty} \langle t \rangle \|\partial_t \tilde{S} \Gamma^\beta \mathbf{v}\|_{L^\infty} \\ &\quad + \sum_{\beta+\gamma=\alpha} \|\langle r \rangle \Gamma^\gamma \eta\|_{L^\infty} \|\langle r \rangle \langle t-r \rangle \partial_r \partial_t \Gamma^\beta \mathbf{v}\|_{L^\infty} \\ &\lesssim \delta \mathcal{E}_{[\kappa/2]+4}^{\frac{1}{2}}(t). \end{aligned}$$

This completes the proof. $\square$

3. Energy estimates

This section is devoted to the energy estimates. For any integer $\kappa \geq 12$ and multi-index $\alpha \in \mathbb{N}^8$ satisfying $|\alpha| \leq \kappa - 1$, we apply Lemma 1 to the system (2.4)-(2.5) to get

$$\begin{aligned} &\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\partial \Gamma^\alpha \mathbf{v}|^2(t) dx + L\mathcal{E}_\kappa(T) \\ &\leq C_0 \int_{\mathbb{R}^3} |\partial \Gamma^\alpha \mathbf{v}|^2(0) dx + C_0 \int_0^T \int_{\mathbb{R}^3} \left[\left(|\partial \eta| + \frac{|\eta|}{r^{\frac{1}{2}} \langle r \rangle^{\frac{1}{2}}} \right) |\partial \Gamma^\alpha \mathbf{v}| \left(|\partial \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right] dx dt \\ &\quad + C_0 \left| \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^\alpha \mathbf{v} \cdot N^\alpha dx dt \right| + C_0 \sup_{k \geq 0} \left| \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot N^\alpha dx dt \right|. \end{aligned} \quad (3.1)$$

For the second term on the right-hand side of (3.1), by Lemma 2, one has

$$\begin{aligned} &C_0 \int_0^T \int_{\mathbb{R}^3} \left[\left(|\partial \eta| + \frac{|\eta|}{r^{\frac{1}{2}} \langle r \rangle^{\frac{1}{2}}} \right) |\partial \Gamma^\alpha \mathbf{v}| \left(|\partial \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right] dx dt \\ &\lesssim \int_0^T \left(\|\langle r \rangle r \partial \eta\|_{L^\infty} + \|\langle r \rangle \eta\|_{L^\infty} \right) \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} |\partial \Gamma^\alpha \mathbf{v}| \right\|_{L^2} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} dt \\ &\lesssim \sum_{i,j,k,l \leq 1} \int_0^T \|r^i \partial_r^j \tilde{\Omega}^l \partial^k \eta\|_{L^2} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} |\partial \Gamma^\alpha \mathbf{v}| \right\|_{L^2} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} dt \\ &\lesssim \delta L\mathcal{E}_\kappa(T). \end{aligned}$$

By utilizing (2.4), we formulate the third term on the right-hand side of (3.1) as follows

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^\alpha \mathbf{v} \cdot N^\alpha dx dt \\ &= - \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^\alpha \mathbf{v} \cdot (\nabla X)^{-T} \nabla \Gamma^\alpha p dx dt - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^\alpha \mathbf{v} \cdot \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} dx dt \end{aligned}$$

$$- \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_{\alpha}^{\beta} \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^{\alpha} \mathbf{v} \cdot [(\nabla X)^{-T} (\nabla \Gamma^{\gamma} \mathbf{v})^T (\partial_t^2 \Gamma^{\beta} \mathbf{v} - \Delta \Gamma^{\beta} \mathbf{v} + \sum_{\iota_1 + \iota_2 = \beta} C_{\beta}^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})] dx dt. \quad (3.2)$$

We handle term by term on the right-hand side of (3.2). For the first term on the right-hand side of (3.2), since $(\nabla X)^{-T}$ is composed of elements of the form $\partial_l \mathbf{v}_j \partial_m \mathbf{v}_k$, $\partial_j \mathbf{v}_j$, $\partial_i \mathbf{v}_j$ and the constant 1, where $j, k, l, m = 1, 2, 3$, it follows that

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^{\alpha} \mathbf{v} \cdot (\nabla X)^{-T} \nabla \Gamma^{\alpha} p dx dt \\ & \lesssim \sum_{\substack{i,j,k,l,m,n=1,2,3 \\ i \neq j \neq k}} \int_0^T \int_{\mathbb{R}^3} |\partial_t \Gamma^{\alpha} \mathbf{v}_i \partial_l \mathbf{v}_j \partial_m \mathbf{v}_k \partial_n \Gamma^{\alpha} p| dx dt + \sum_{\substack{i,j=1,2,3 \\ i \neq j}} \int_0^T \int_{\mathbb{R}^3} |\partial_t \Gamma^{\alpha} \mathbf{v}_i \partial_j \mathbf{v}_j \partial_i \Gamma^{\alpha} p| dx dt \\ & + \sum_{\substack{i,j=1,2,3 \\ i \neq j}} \int_0^T \int_{\mathbb{R}^3} |\partial_t \Gamma^{\alpha} \mathbf{v}_i \partial_i \mathbf{v}_j \partial_j \Gamma^{\alpha} p| dx dt + \left| \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^{\alpha} \mathbf{v} \cdot \nabla \Gamma^{\alpha} p dx dt \right|. \end{aligned} \quad (3.3)$$

For the first term on the right-hand side of (3.3), by (2.6) and Lemma 6, we have

$$\begin{aligned} & - \sum_{\substack{i,j,k,l,m,n=1,2,3 \\ i \neq j \neq k}} \int_0^T \int_{\mathbb{R}^3} |\partial_t \Gamma^{\alpha} \mathbf{v}_i \partial_l \mathbf{v}_j \partial_m \mathbf{v}_k \partial_n \Gamma^{\alpha} p| dx dt \lesssim \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^{\alpha} \mathbf{v}\|_{L^2} \langle t \rangle^2 \|\nabla \mathbf{v}\|_{L^{\infty}}^2 \|\nabla \Gamma^{\alpha} p\|_{L^2} dt \\ & \lesssim \int_0^T \langle t \rangle^{-2} \mathcal{E}_{\kappa}(t) \mathcal{E}_{[\kappa/2]+2}(t) dt. \end{aligned}$$

To handle the second and third terms on the right-hand side of (3.3), we apply (2.11) and (2.19) to show

$$\begin{aligned} & \sum_{\substack{i,j=1,2,3 \\ i \neq j}} \int_0^T \int_{\mathbb{R}^3} |\partial_t \Gamma^{\alpha} \mathbf{v}_i \partial_j \mathbf{v}_j \partial_i \Gamma^{\alpha} p| dx dt + \sum_{\substack{i,j=1,2,3 \\ i \neq j}} \int_0^T \int_{\mathbb{R}^3} |\partial_t \Gamma^{\alpha} \mathbf{v}_i \partial_i \mathbf{v}_j \partial_j \Gamma^{\alpha} p| dx dt \\ & \lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \|\partial_t \Gamma^{\alpha} \mathbf{v}\|_{L^2(\mathcal{R})} \langle t \rangle^{\frac{3}{2}} \|\nabla \mathbf{v}\|_{L^{\infty}(\mathcal{R})} \|\nabla \Gamma^{\alpha} p\|_{L^2(\mathcal{R})} dt \\ & + \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^{\alpha} \mathbf{v}\|_{L^2(\mathcal{R}^c)} \|\langle r \rangle (1 - \xi(s)) \nabla \mathbf{v}\|_{L^{\infty}(\mathcal{R}^c)} \langle t \rangle \|\nabla \Gamma^{\alpha} p\|_{L^2(\mathcal{R}^c)} dt \\ & \lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_{\kappa}^{\frac{3}{2}}(t) dt. \end{aligned}$$

Along the same line, the last term on the right-hand side of (3.3) is estimated by

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^{\alpha} \mathbf{v} \cdot \nabla \Gamma^{\alpha} p dx dt \right| = \left| \int_0^T \int_{\mathbb{R}^3} \partial_t (\nabla \cdot \Gamma^{\alpha} \mathbf{v}) \cdot \Gamma^{\alpha} p dx dt \right| \\ & = \left| \int_0^T \int_{\mathbb{R}^3} \nabla \cdot \left[\sum_{\beta+\gamma+\iota=\alpha} \partial_t \Gamma^{\beta} \mathbf{v}_1 \begin{pmatrix} \partial_2 \Gamma^{\gamma} \mathbf{v}_2 \partial_3 \Gamma^{\iota} \mathbf{v}_3 - \partial_3 \Gamma^{\gamma} \mathbf{v}_2 \partial_2 \Gamma^{\iota} \mathbf{v}_3 \\ \partial_3 \Gamma^{\gamma} \mathbf{v}_2 \partial_1 \Gamma^{\iota} \mathbf{v}_3 - \partial_1 \Gamma^{\gamma} \mathbf{v}_2 \partial_3 \Gamma^{\iota} \mathbf{v}_3 \\ \partial_1 \Gamma^{\gamma} \mathbf{v}_2 \partial_2 \Gamma^{\iota} \mathbf{v}_3 - \partial_2 \Gamma^{\gamma} \mathbf{v}_2 \partial_1 \Gamma^{\iota} \mathbf{v}_3 \end{pmatrix} \right. \right. \\ & \quad \left. \left. - \sum_{\beta+\gamma=\alpha} \begin{pmatrix} \partial_t \Gamma^{\beta} \mathbf{v}_1 \partial_2 \Gamma^{\gamma} \mathbf{v}_2 + \partial_t \Gamma^{\beta} \mathbf{v}_1 \partial_3 \Gamma^{\gamma} \mathbf{v}_3 - \partial_2 \Gamma^{\beta} \mathbf{v}_1 \partial_t \Gamma^{\gamma} \mathbf{v}_2 - \partial_3 \Gamma^{\beta} \mathbf{v}_1 \partial_t \Gamma^{\gamma} \mathbf{v}_3 \\ -\partial_t \Gamma^{\beta} \mathbf{v}_1 \partial_1 \Gamma^{\gamma} \mathbf{v}_2 + \partial_t \Gamma^{\beta} \mathbf{v}_2 \partial_3 \Gamma^{\gamma} \mathbf{v}_3 + \partial_1 \Gamma^{\beta} \mathbf{v}_1 \partial_t \Gamma^{\gamma} \mathbf{v}_2 - \partial_3 \Gamma^{\beta} \mathbf{v}_2 \partial_t \Gamma^{\gamma} \mathbf{v}_3 \\ -\partial_t \Gamma^{\beta} \mathbf{v}_2 \partial_2 \Gamma^{\gamma} \mathbf{v}_3 - \partial_t \Gamma^{\beta} \mathbf{v}_1 \partial_1 \Gamma^{\gamma} \mathbf{v}_3 + \partial_2 \Gamma^{\beta} \mathbf{v}_2 \partial_t \Gamma^{\gamma} \mathbf{v}_3 + \partial_1 \Gamma^{\beta} \mathbf{v}_1 \partial_t \Gamma^{\gamma} \mathbf{v}_3 \end{pmatrix} \right] \Gamma^{\alpha} p dx dt \right| \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \leq |\beta|}} \int_0^T \|\partial \Gamma^\beta \mathbf{v}\|_{L^2} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} (1 + \|\partial \Gamma^\iota \mathbf{v}\|_{L^\infty}) \|\nabla \Gamma^\alpha p\|_{L^2} dt \\
&\lesssim \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \leq |\beta|}} \int_0^T \langle t \rangle^{-\frac{3}{2}} \|\partial \Gamma^\beta \mathbf{v}\|_{L^2} (1 + \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty}) \left(\langle t \rangle^{\frac{3}{2}} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} \|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R})} \right. \\
&\quad \left. + \|\langle r \rangle (1 - \xi(s)) \partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \langle t \rangle \|\nabla \Gamma^\alpha p\|_{L^2(\mathcal{R}^c)} \right) dt \\
&\lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_\kappa(t) \mathcal{E}_{[\kappa/2]+3}^{\frac{1}{2}}(t) dt.
\end{aligned}$$

For the second term on the right-hand side of (3.2), by the definition of $\widetilde{S}$ and (2.6), one has

$$\begin{aligned}
& - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_\alpha^\beta \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^\alpha \mathbf{v} \cdot \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v} dx dt \\
&\lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-1} \|r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \partial_t \Gamma^\alpha \mathbf{v}\|_{L^2} \|\langle r \rangle \Gamma^\gamma \eta\|_{L^\infty} \|\partial_t \widetilde{S} \Gamma^\beta \mathbf{v}\|_{L^2} dt \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^\alpha \mathbf{v}\|_{L^2} \|\langle r \rangle^2 \Gamma^\gamma \eta\|_{L^\infty} \left(\|\langle t-r \rangle \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} + \|\langle t-r \rangle \partial_t \partial_r \Gamma^\beta \mathbf{v}\|_{L^2} \right) dt \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^\alpha \mathbf{v}\|_{L^2} \|\langle r \rangle \Gamma^\gamma \eta\|_{L^2} \left(\langle t \rangle \|\partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} + \langle t \rangle \|\partial_t \widetilde{S} \Gamma^\beta \mathbf{v}\|_{L^\infty} + \langle t \rangle \|\partial_t \partial_r \Gamma^\beta \mathbf{v}\|_{L^\infty} \right) dt \\
&\lesssim \delta L \mathcal{E}_\kappa(T) + \delta \int_0^T \langle t \rangle^{-2} \mathcal{E}_\kappa(t) dt.
\end{aligned}$$

It is left to estimate the last term on the right-hand side of (3.2). Two cases are considered. By Lemmas 2, 4, and (2.6), we solve the case $|\gamma| < |\beta| < |\alpha|$ by

$$\begin{aligned}
& - \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} C_\alpha^\beta \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^\alpha \mathbf{v} \cdot [(\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})] dx dt \\
&\lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^\alpha \mathbf{v}\|_{L^2} \langle t \rangle \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty} \langle t \rangle \|\rho \partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v}\|_{L^2} dt \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha, \iota_1+\iota_2=\beta \\ |\gamma| < |\beta| < |\alpha|, |\iota_1| \leq |\iota_2|}} \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^\alpha \mathbf{v}\|_{L^2} \langle t \rangle \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\Gamma^{\iota_2} \eta\|_{L^2} \|\langle r \rangle \langle t-r \rangle \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^\infty} dt \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha, \iota_1+\iota_2=\beta \\ |\gamma| < |\beta| < |\alpha|, |\iota_2| < |\iota_1|}} \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^\alpha \mathbf{v}\|_{L^2} \langle t \rangle \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\langle r \rangle \Gamma^{\iota_2} \eta\|_{L^\infty} \|\langle t-r \rangle \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^2} dt \\
&\lesssim \int_0^T \langle t \rangle^{-2} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt.
\end{aligned}$$

Utilizing Lemmas 9 and 10, we solve the case $|\beta| \leq |\gamma|$ by

$$\begin{aligned}
& - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} C_{\alpha}^{\beta} \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^{\alpha} \mathbf{v} \cdot [(\nabla X)^{-T} (\nabla \Gamma^{\gamma} \mathbf{v})^T (\partial_t^2 \Gamma^{\beta} \mathbf{v} - \Delta \Gamma^{\beta} \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_{\beta}^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})] dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^{\alpha} \mathbf{v}\|_{L^2} \|\nabla \Gamma^{\gamma} \mathbf{v}\|_{L^2} \langle t \rangle^2 \|\rho \partial_t^2 \Gamma^{\beta} \mathbf{v} - \Delta \Gamma^{\beta} \mathbf{v}\|_{L^{\infty}} dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha, \iota_1+\iota_2=\beta \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-2} \|\partial_t \Gamma^{\alpha} \mathbf{v}\|_{L^2} \|\nabla \Gamma^{\gamma} \mathbf{v}\|_{L^2} \langle t \rangle^2 \|\Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^{\infty}} dt \\
& \lesssim \int_0^T \langle t \rangle^{-2} \mathcal{E}_{\kappa}^{\frac{3}{2}}(t) dt.
\end{aligned}$$

By summing up the above estimates, we deduce that

$$\left| \int_0^T \int_{\mathbb{R}^3} \partial_t \Gamma^{\alpha} \mathbf{v} \cdot N^{\alpha} dx dt \right| \lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_{\kappa}^{\frac{3}{2}}(t) dt + \delta \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_{\kappa}(t) dt + \delta L \mathcal{E}_{\kappa}(T).$$

We continue to handle the last term on the right-hand side of (3.1). The identity (2.4) yields that

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^{\alpha} \mathbf{v} + \frac{\Gamma^{\alpha} \mathbf{v}}{r} \right) \cdot N^{\alpha} dx dt \\
& = - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_{\alpha}^{\beta} \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^{\alpha} \mathbf{v} + \frac{\Gamma^{\alpha} \mathbf{v}}{r} \right) \cdot \Gamma^{\gamma} \eta \partial_t^2 \Gamma^{\beta} \mathbf{v} dx dt \\
& \quad - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} C_{\alpha}^{\beta} \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^{\alpha} \mathbf{v} + \frac{\Gamma^{\alpha} \mathbf{v}}{r} \right) \cdot [(\nabla X)^{-T} (\nabla \Gamma^{\gamma} \mathbf{v})^T (\partial_t^2 \Gamma^{\beta} \mathbf{v} - \Delta \Gamma^{\beta} \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_{\beta}^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})] dx dt \\
& \quad - \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^{\alpha} \mathbf{v} + \frac{\Gamma^{\alpha} \mathbf{v}}{r} \right) \cdot (\nabla X)^{-T} \nabla \Gamma^{\alpha} p dx dt. \tag{3.4}
\end{aligned}$$

We use Lemmas 9 and 10 to handle the first two terms on the right-hand side of (3.4) by

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^{\alpha} \mathbf{v}| + \frac{|\Gamma^{\alpha} \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle^2 \Gamma^{\gamma} \eta\|_{L^{\infty}} \|\langle t-r \rangle \partial_t^2 \Gamma^{\beta} \mathbf{v}\|_{L^2} dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^{\alpha} \mathbf{v}| + \frac{|\Gamma^{\alpha} \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \Gamma^{\gamma} \eta\|_{L^2} \|\langle r \rangle \langle t-r \rangle \partial_t^2 \Gamma^{\beta} \mathbf{v}\|_{L^{\infty}} dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-2} \left\| |\partial_r \Gamma^{\alpha} \mathbf{v}| + \frac{|\Gamma^{\alpha} \mathbf{v}|}{r} \right\|_{L^2} \|\nabla \Gamma^{\gamma} \mathbf{v}\|_{L^{\infty}} \left(\langle t \rangle \|\partial_t^2 \Gamma^{\beta} \mathbf{v} - \Delta \Gamma^{\beta} \mathbf{v}\|_{L^2} \right. \\
& \quad \left. + \sum_{\iota_1+\iota_2=\beta} \|\langle r \rangle \langle t-r \rangle \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^2} \right) dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-2} \left\| |\partial_r \Gamma^{\alpha} \mathbf{v}| + \frac{|\Gamma^{\alpha} \mathbf{v}|}{r} \right\|_{L^2} \|\nabla \Gamma^{\gamma} \mathbf{v}\|_{L^2} \left(\langle t \rangle^2 \|\rho \partial_t^2 \Gamma^{\beta} \mathbf{v} - \Delta \Gamma^{\beta} \mathbf{v}\|_{L^{\infty}} \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\iota_1 + \iota_2 = \beta} \langle t \rangle^2 \|\Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}\|_{L^\infty} \rangle dt \\
& \lesssim \delta L \mathcal{E}_\kappa(T) + \delta \int_0^T \langle t \rangle^{-2} \mathcal{E}_\kappa(t) dt + \int_0^T \langle t \rangle^{-2} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt.
\end{aligned}$$

We rewrite the last term on the right-hand side of (3.4) as follows

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot (\nabla X)^{-T} \nabla \Gamma^\alpha p \, dx dt \\
& \lesssim \sum_{\substack{i,j=1,2,3 \\ i \neq j}} \int_0^T \int_{\mathbb{R}^3} \left(\left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v}_i + \frac{\Gamma^\alpha \mathbf{v}_i}{r} \right) \partial_j \mathbf{v}_j \partial_i \Gamma^\alpha p \right| + \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v}_i + \frac{\Gamma^\alpha \mathbf{v}_i}{r} \right) \partial_i \mathbf{v}_j \partial_j \Gamma^\alpha p \right| \right) dx dt \\
& + \sum_{\substack{j,l,m,n,s=1,2,3 \\ i \neq j \neq l}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v}_i + \frac{\Gamma^\alpha \mathbf{v}_i}{r} \right) \partial_m \mathbf{v}_j \partial_n \mathbf{v}_l \partial_s \Gamma^\alpha p \right| dx dt + \left| \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Gamma^\alpha p \, dx dt \right|.
\end{aligned} \tag{3.5}$$

For the first two terms on the right-hand side of (3.5), the Lemma 6 and (2.19) imply that

$$\begin{aligned}
& \int_0^T \langle t \rangle^{-2} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right\|_{L^2} (1 + \|\nabla \mathbf{v}\|_{L^\infty}) \|\langle r \rangle (1 - \xi(s)) \nabla \mathbf{v}\|_{L^\infty} \langle t \rangle \|\nabla \Gamma^\alpha p\|_{L^2(\mathbb{R}^c)} \, dt \\
& + \int_0^T \langle t \rangle^{-\frac{3}{2}} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right\|_{L^2} (1 + \|\nabla \mathbf{v}\|_{L^\infty}) \langle t \rangle^{\frac{3}{2}} \|\nabla \mathbf{v}\|_{L^\infty(\mathbb{R})} \|\nabla \Gamma^\alpha p\|_{L^2(\mathbb{R})} \, dt \\
& \lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_\kappa^{\frac{3}{2}}(t) \, dt.
\end{aligned}$$

We use (2.7) to formulate the last term on the right-hand side of (3.5) by

$$\begin{aligned}
& \left| \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Gamma^\alpha p \, dx dt \right| \\
& \lesssim \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot [(\nabla \mathbf{v})^T (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})] \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot [(\nabla X)^T \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}] \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot [(\nabla \Gamma^\gamma \mathbf{v})^T (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1 + \iota_2 = \beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})] \right| dx dt \\
& + \left| \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}) \, dx dt \right|.
\end{aligned} \tag{3.6}$$

By (2.19) and Lemma 6, we estimate the first term on the right-hand side of (3.6) by

$$\int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot [(\nabla \mathbf{v})^T (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v})] \right| dx dt$$

$$\begin{aligned}
&\lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \langle t \rangle^{\frac{3}{2}} \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{R})} \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} dt \\
&\quad + \int_0^T \langle t \rangle^{-2} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \|\langle r \rangle (1 - \xi(s)) \nabla \mathbf{v}\|_{L^\infty} \langle t \rangle \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2(\mathcal{R})} dt \\
&\lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt.
\end{aligned} \tag{3.7}$$

The second and third terms on the right-hand side of (3.6) can be solved using the same method employed by the first two terms of (3.4).

For the last term on the right-hand side of (3.6), we observe that

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot (\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}) dx dt \right| \\
&\lesssim \left| \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} (\rho \partial_t^2 \nabla \cdot \Gamma^\alpha \mathbf{v} - \Delta \nabla \cdot \Gamma^\alpha \mathbf{v}) dx dt \right| \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} (\nabla \eta \cdot \partial_t^2 \Gamma^\alpha \mathbf{v}) \right| dx dt.
\end{aligned} \tag{3.8}$$

In view of (2.5), we write the first term on the right-hand side of (3.8) by

$$\begin{aligned}
&\sum_{\substack{i,j=1,2,3 \\ i \neq j}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot [(\rho \partial_t^2 \Gamma^\alpha \mathbf{v}_i - \Delta \Gamma^\alpha \mathbf{v}_i) \partial \mathbf{v}_j] \right| dx dt \\
&+ \sum_{\substack{i,j,l=1,2,3 \\ i \neq j \neq l}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot [(\rho \partial_t^2 \Gamma^\alpha \mathbf{v}_i - \Delta \Gamma^\alpha \mathbf{v}_i) \partial \mathbf{v}_j \partial \mathbf{v}_l] \right| dx dt \\
&+ \sum_{\substack{\beta+\gamma+i=\alpha, |\beta| \neq |\alpha| \\ i,j,l=1,2,3, i \neq j \neq l}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot (\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \partial \Gamma^\alpha \mathbf{v}_l) \right| dx dt \\
&+ \sum_{\substack{\beta+\gamma+i=\alpha, |\beta| \neq |\alpha| \\ i,j,l=1,2,3, i \neq j \neq m}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} (\partial \eta \partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \partial \Gamma^\alpha \mathbf{v}_l) \right| dx dt \\
&+ \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i,j=1,2,3, i \neq j}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot (\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j) \right| dx dt \\
&+ \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i,j=1,2,3, i \neq j}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} (\partial \eta \partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j) \right| dx dt.
\end{aligned} \tag{3.9}$$

The first two terms on the right-hand side of (3.9) are dealt with by the same method as (3.7). By (2.6), the third and fourth terms on the right-hand side of (3.9) are estimated by

$$\int_0^T \langle t \rangle^{-2} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} (1 + \|\nabla \eta\|_{L^3}) \left(\sum_{\substack{\beta+\gamma+i=\alpha \\ |\gamma|, |i| < |\beta| < |\alpha|}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \langle t \rangle \|\partial \Gamma^\alpha \mathbf{v}\|_{L^\infty} \right)$$

$$\begin{aligned}
& + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\iota| \leq |\gamma|}} \langle t \rangle \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} \langle t \rangle \|\partial \Gamma^\iota \mathbf{v}\|_{L^\infty} + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\gamma| \leq |\iota|}} \langle t \rangle \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \langle t \rangle \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\iota \mathbf{v}\|_{L^2} \Big) dt \\
& \lesssim \int_0^T \langle t \rangle^{-2} \mathcal{E}_k^{\frac{3}{2}}(t) dt.
\end{aligned}$$

To consider the fifth term on the right-hand side of (3.9), we separate two cases to consider it. For the case $x \in \mathcal{R}$, by (2.20), we have

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i, j=1, 2, 3, i \neq j}} \left| \int_0^T \int_{\mathbb{R}^3} \xi(s) f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot \left(\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \right) dx dt \right| \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i, j=1, 2, 3, i \neq j}} \int_0^T \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j\|_{L^2} dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \left(\|\langle r \rangle \langle t-r \rangle (1-\xi(s)) \partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \right. \\
& \quad \left. + \langle t \rangle^2 \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty(\mathcal{R})} \right) \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} dt \\
& \lesssim \sup_{0 \leq t \leq T} \mathcal{E}_k^{\frac{1}{2}}(t) L \mathcal{E}_k(T) + \int_0^T \langle t \rangle^{-2} \mathcal{E}_k^{\frac{3}{2}}(t) dt.
\end{aligned}$$

To consider the case $x \in \mathcal{R}^c$, we use the integration by parts to get

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i, j=1, 2, 3, i \neq j}} \left| \int_0^T \int_{\mathbb{R}^3} [1-\xi(s)] f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \cdot \left(\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \right) dx dt \right| \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i, j=1, 2, 3, i \neq j}} \left| \int_0^T \int_{\mathbb{R}^3} \nabla \cdot \left[[1-\xi(s)] f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \right] \cdot \Delta^{-1} \nabla \cdot \left(\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \right) dx dt \right| \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i, j=1, 2, 3, i \neq j}} \int_0^T \langle t \rangle^{-1} \left\| |\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \left\| \Delta^{-1} \nabla \cdot \left(\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \right) \right\|_{L^2} dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i, j=1, 2, 3, i \neq j}} \int_0^T \|\nabla \cdot \Gamma^\alpha \mathbf{v}\|_{L^2} \|\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j\|_{L^2} dt. \tag{3.10}
\end{aligned}$$

For the first term on the right-hand side of (3.10), we have

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i, j=1, 2, 3, i \neq j}} \int_0^T \langle t \rangle^{-1} \left\| |\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \left\| \Delta^{-1} \nabla \cdot \left(\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \right) \right\|_{L^2} dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-\frac{4}{3}} \left\| |\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \langle t \rangle^{\frac{1}{3}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty}^{\frac{1}{3}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2}^{\frac{2}{3}} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} dt
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \int_0^T \langle t \rangle^{-\frac{4}{3}} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2}^{\frac{2}{3}} \langle t \rangle^{\frac{1}{3}} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty}^{\frac{1}{3}} dt \\
& \lesssim \int_0^T \langle t \rangle^{-\frac{4}{3}} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt.
\end{aligned}$$

We use (2.5) to estimate the second term on the right-hand side of (3.10) by

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i,j=1,2,3, i \neq j}} \int_0^T \|\nabla \cdot \Gamma^\alpha \mathbf{v}\|_{L^2} \|\partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j\|_{L^2} dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha, \beta+\tilde{\gamma}+\tilde{i}=\alpha \\ |\beta| \leq |\gamma|, |\tilde{\gamma}|, |\tilde{i}| \leq |\beta|}} \int_0^T \langle t \rangle^{-2} \|\partial \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\partial \Gamma^{\tilde{\gamma}} \mathbf{v}\|_{L^\infty} (1 + \|\partial \Gamma^{\tilde{i}} \mathbf{v}\|_{L^\infty}) \langle t \rangle \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, \beta+\tilde{\gamma}+\tilde{i}=\alpha \\ |\gamma|<|\beta|<|\alpha|, |\tilde{\gamma}|, |\tilde{i}| \leq |\beta|}} \int_0^T \langle t \rangle^{-2} \|\partial \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\partial \Gamma^{\tilde{\gamma}} \mathbf{v}\|_{L^\infty} (1 + \|\partial \Gamma^{\tilde{i}} \mathbf{v}\|_{L^\infty}) \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} dt \\
& \lesssim \int_0^T \langle t \rangle^{-2} \mathcal{E}_\kappa^2(t) dt.
\end{aligned}$$

For the last term on the right-hand side of (3.9), it follows from the Sobolev embedding inequality, (2.6), and (2.19), that

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha, |\beta| \neq |\alpha| \\ i,j=1,2,3, i \neq j}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\partial \eta \partial^2 \Gamma^\beta \mathbf{v}_i \partial \Gamma^\gamma \mathbf{v}_j \right) \right| dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|<|\alpha|}} \int_0^T \langle t \rangle^{-2} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^3} \|\langle t-r \rangle \partial^2 \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\partial \Gamma^\gamma \mathbf{v}\|_{L^\infty} dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-\frac{3}{2}} \left\| \partial_r \Gamma^\alpha \mathbf{v} + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \|\partial \Gamma^\gamma \mathbf{v}\|_{L^2} \left(\|\nabla \eta\|_{L^3(\mathbb{R})} \langle t \rangle^{\frac{3}{2}} \|\partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty(\mathbb{R})} \right. \\
& \quad \left. + \|\langle r \rangle (1 - \xi(s)) \nabla \eta\|_{L^3} \|\langle r \rangle (1 - \xi(s)) \partial^2 \Gamma^\beta \mathbf{v}\|_{L^\infty} \right) dt \\
& \lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt.
\end{aligned}$$

To solve the second term on the right-hand side of (3.8), we rewrite it as follows

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \Gamma^\alpha \mathbf{v} \right) \right| dx dt \\
& \lesssim \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \nabla \Delta^{-1} \nabla \cdot \Gamma^\alpha \mathbf{v} \right) \right| dx dt \\
& + \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \nabla^\perp \Delta^{-1} \nabla^\perp \cdot \Gamma^\alpha \mathbf{v} \right) \right| dx dt. \tag{3.11}
\end{aligned}$$

In view of (2.5), we formulate the first term on the right-hand side of (3.11) by

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \nabla \Delta^{-1} \nabla \cdot \Gamma^\alpha \mathbf{v} \right) \right| dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \nabla \Delta^{-1} (\partial_l \Gamma^\beta \mathbf{v}_i \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ i,j,l=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \nabla \Delta^{-1} (\partial_l \Gamma^\beta \mathbf{v}_1 \partial_j \Gamma^\gamma \mathbf{v}_2 \partial_l \Gamma^\iota \mathbf{v}_3) \right) \right| dx dt. \quad (3.12)
\end{aligned}$$

Since the second term has analogous estimates to the first term of (3.12), it suffices to concentrate on the first term. We observe that

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \nabla \Delta^{-1} (\partial_l \Gamma^\beta \mathbf{v}_i \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\partial_t^2 \partial_l \Gamma^\beta \mathbf{v}_i \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\partial_t \partial_l \Gamma^\beta \mathbf{v}_i \partial_t \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\partial_l \Gamma^\beta \mathbf{v}_i \partial_t^2 \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt. \quad (3.13)
\end{aligned}$$

Here, we restrict our analysis to the first term on the right-hand side of (3.13), as the remaining two terms have similar estimates.

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\partial_t^2 \partial_l \Gamma^\beta \mathbf{v}_i \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\gamma| < |\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\partial_t^2 \Gamma^\beta \mathbf{v}_i \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, |\gamma| < |\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} \partial_l (\partial_t^2 \Gamma^\beta \mathbf{v}_i \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, |\beta| \leq |\gamma| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\partial_t^2 \partial_l \Gamma^\beta \mathbf{v}_i \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt. \quad (3.14)
\end{aligned}$$

For the first term on the right-hand side of (3.14), we deduce from (2.4) that

$$\sum_{\substack{\beta+\gamma=\alpha, |\gamma| < |\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\partial_t^2 \Gamma^\beta \mathbf{v}_i \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt$$

$$\begin{aligned}
& \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\gamma| \leq |\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\rho^{-1} \Delta \Gamma^\beta \mathbf{v}_i \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, |\gamma| \leq |\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\rho^{-1} N_i^\beta \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt. \quad (3.15)
\end{aligned}$$

We formulate the first term on the right-hand side of (3.15) as follows

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha, |\gamma| \leq |\beta| \\ i,j,l,m,n=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \left(\eta \nabla \Delta^{-1} \partial_n (\rho^{-1} \partial_n \Gamma^\beta \mathbf{v}_i \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, |\gamma| \leq |\beta| \\ i,j,l,m,n=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \partial_n \left(\rho^{-1} \eta \partial_n \Gamma^\beta \mathbf{v}_i \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, |\gamma| \leq |\beta| \\ i,j,l,m,n=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \left(\eta \nabla \Delta^{-1} [\partial_n \Gamma^\beta \mathbf{v}_i \partial_n (\rho^{-1} \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j)] \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, |\gamma| \leq |\beta| \\ i,j,l,m,n=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\rho^{-1} \partial_n \eta \partial_n \Gamma^\beta \mathbf{v}_i \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha, |\gamma| \leq |\beta| \\ i,j,l,m,n=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\eta \partial_n \Gamma^\beta \mathbf{v}_i \partial_n (\rho^{-1} \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt. \quad (3.16)
\end{aligned}$$

The first three terms on the right-hand side of (3.16) can be estimated by

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| \leq |\beta|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \eta\|_{L^\infty} \|\nabla \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\nabla^2 \Gamma^\gamma \mathbf{v}\|_{L^\infty} dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| \leq |\beta|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \eta\|_{L^3} \|\nabla \Gamma^\beta \mathbf{v}\|_{L^2} \langle t \rangle \|\nabla (\rho^{-1} \nabla^2 \Gamma^\gamma \mathbf{v})\|_{L^\infty} dt \\
& \lesssim \delta L \mathcal{E}_\kappa(T) + \int_0^T \langle t \rangle^{-2} \mathcal{E}_\kappa^2(t) dt.
\end{aligned}$$

We utilize (2.19) to solve the last two terms on the right-hand side of (3.16) by

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| \leq |\beta|}} \int_0^T \langle t \rangle^{-\frac{3}{2}} \left\| |\partial_r \Gamma^\beta \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \|\nabla \Gamma^\beta \mathbf{v}\|_{L^2} \left(\|\langle r \rangle (1 - \xi(s)) \nabla \eta\|_{L^3} \|\langle r \rangle (1 - \xi(s)) \nabla^2 \Gamma^\gamma \mathbf{v}\|_{L^\infty} \right. \\
& + \|\nabla \eta\|_{L^3(\mathcal{R})} \langle t \rangle^{\frac{3}{2}} \|\nabla^2 \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} + \|\langle r \rangle (1 - \xi(s)) \eta\|_{L^3} \|\langle r \rangle (1 - \xi(s)) \nabla (\rho^{-1} \nabla^2 \Gamma^\gamma \mathbf{v})\|_{L^\infty} \\
& \left. + \|\eta\|_{L^3(\mathcal{R})} \langle t \rangle^{\frac{3}{2}} \|\nabla (\rho^{-1} \nabla^2 \Gamma^\gamma \mathbf{v}_j)\|_{L^\infty(\mathcal{R})} \right) dt \\
& \lesssim \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt.
\end{aligned}$$

Along the same line, the second term on the right-hand side of (3.15) can be handled by

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha, |\gamma|<|\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \nabla \Delta^{-1} (\rho^{-1} N_i^\beta \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha, |\gamma|<|\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla \left(\eta \cdot \nabla \Delta^{-1} (\rho^{-1} N_i^\beta \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right) \right| dx dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha, |\gamma|<|\beta| \\ i,j,l,m=1,2,3}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} (\rho^{-1} \eta N_i^\beta \partial_l \partial_m \Gamma^\gamma \mathbf{v}_j) \right| dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \eta\|_{L^3} \|N^\beta\|_{L^2} \langle t \rangle \|\nabla^2 \Gamma^\gamma \mathbf{v}\|_{L^\infty} dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|}} \int_0^T \langle t \rangle^{-2} \left\| |\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \|N^\beta\|_{L^2} \|\langle r \rangle (1 - \xi(s)) \eta\|_{L^3} \|\langle r \rangle (1 - \xi(s)) \nabla^2 \Gamma^\gamma \mathbf{v}\|_{L^\infty} dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<|\beta|}} \int_0^T \langle t \rangle^{-\frac{3}{2}} \left\| |\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right\|_{L^2} \|N^\beta\|_{L^2} \|\eta\|_{L^3(\mathcal{R})} \langle t \rangle^{\frac{3}{2}} \|\nabla^2 \Gamma^\gamma \mathbf{v}\|_{L^\infty(\mathcal{R})} dt \\
& \lesssim \delta L \mathcal{E}_\kappa(T) + \int_0^T \langle t \rangle^{-\frac{3}{2}} \mathcal{E}_\kappa^2(t) dt.
\end{aligned}$$

The same estimates hold for the last two terms on the right-hand side of (3.14).

Applying (2.4), we formulate the second term on the right-hand side of (3.11) as follows

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \left(\nabla \eta \cdot \partial_t^2 \nabla^\perp \Delta^{-1} \nabla^\perp \cdot \Gamma^\alpha \mathbf{v} \right) \right| dx dt \\
& = \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \cdot \partial_t^2 \Delta^{-1} \nabla^\perp \cdot \Gamma^\alpha \mathbf{v} \right) \right| dx dt \\
& \lesssim \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \nabla^\perp \cdot \Gamma^\alpha \mathbf{v} \right) \right| dx dt \\
& \quad + \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot [(\nabla X)^{-T} \nabla \Gamma^\alpha p] \right) \right| dx dt \\
& \quad + \sum_{\beta+\gamma=\alpha} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}) \right) \right| dx dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot [C_\alpha^\beta (\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T \right. \right. \\
& \quad \left. \left. \cdot (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v})] \right) \right| dx dt. \tag{3.17}
\end{aligned}$$

The first term on the right-hand side of (3.17) is solved by

$$\int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \nabla^\perp \cdot \Gamma^\alpha \mathbf{v} \right) \right| dx dt$$

$$\begin{aligned}
&\lesssim \int_0^T \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle^2 \nabla \eta\|_{L^\infty} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} |\nabla \Gamma^\alpha \mathbf{v}| \right\|_{L^2} dt \\
&\lesssim \delta \mathcal{L} \mathcal{E}_k(T).
\end{aligned}$$

The calculations in (2.12) imply that

$$\begin{aligned}
&\|\nabla \Gamma^\alpha p\|_{L^2} + \|\rho \partial_t^2 \Gamma^\alpha \mathbf{v} - \Delta \Gamma^\alpha \mathbf{v}\|_{L^2} \\
&\lesssim \sum_{\beta+\gamma=\alpha} \langle t \rangle^{-1} \|\langle r \rangle \langle t-r \rangle |\partial^2 \Gamma^\beta \mathbf{v}| |\partial \Gamma^\gamma \mathbf{v}|\|_{L^2} + \sum_{\beta+\gamma+\iota=\alpha} \langle t \rangle^{-1} \|\langle r \rangle \langle t-r \rangle |\partial^2 \Gamma^\beta \mathbf{v}| |\partial \Gamma^\gamma \mathbf{v}| |\partial \Gamma^\iota \mathbf{v}|\|_{L^2} \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \langle t \rangle^{-1} \|\langle r \rangle \langle t-r \rangle |\Gamma^\gamma \eta| |\partial_t^2 \Gamma^\beta \mathbf{v}|\|_{L^2} + \sum_{1 \leq \iota \leq 3} \|\langle r \rangle \nabla^\iota \eta\|_{L^\infty} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} |\nabla \Gamma^\alpha \mathbf{v}| \right\|_{L^2} \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \langle t \rangle^{-1} \|\langle r \rangle \nabla \Gamma^\gamma \mathbf{v}\|_{L^\infty} \left\| \langle t-r \rangle (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}) \right\|_{L^2} \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \langle t \rangle^{-1} \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^2} \left\| \langle r \rangle \langle t-r \rangle (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1+\iota_2=\beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}) \right\|_{L^\infty}. \tag{3.18}
\end{aligned}$$

Substituting (3.18) into the second term on the right-hand side of (3.17), we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot [(\nabla X)^{-T} \nabla \Gamma^\alpha p] \right) \right| dx dt \\
&\lesssim \sum_{i,j,l,m,n=1,2,3} \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\partial_l \mathbf{v}_i \partial_m \mathbf{v}_j \partial_n \Gamma^\alpha p) \right) dx dt \\
&\quad + \sum_{i,j,l=1,2,3} \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\partial_j \mathbf{v}_i \partial_l \Gamma^\alpha p) \right) dx dt \\
&\lesssim \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^\infty} \langle t \rangle \|\nabla \mathbf{v}\|_{L^\infty} \|\nabla \mathbf{v}\|_{L^3} \|\nabla \Gamma^\alpha p\|_{L^2} dt \\
&\quad + \int_0^T \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^\infty} \|\nabla \mathbf{v}\|_{L^3} \|\nabla \Gamma^\alpha p\|_{L^2} dt \\
&\lesssim \delta \mathcal{L} \mathcal{E}_k(T) + \int_0^T \langle t \rangle^{-2} \mathcal{E}_k^{\frac{3}{2}}(t) dt. \tag{3.19}
\end{aligned}$$

For the third term on the right-hand side of (3.17), we have

$$\begin{aligned}
&\sum_{\beta+\gamma=\alpha} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}) \right) \right| dx dt \\
&\lesssim \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\eta \partial_t^2 \Gamma^\alpha \mathbf{v}) \right) \right| dx dt \\
&\quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}) \right) \right| dx dt. \tag{3.20}
\end{aligned}$$

We use (2.4) to formulate the first term on the right-hand side of (3.20) as follows

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\rho^{-1} \eta \Delta \Gamma^\alpha \mathbf{v}) \right) \right| dx dt \\
& + \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\rho^{-1} \eta (\nabla X)^{-T} \nabla \Gamma^\alpha p) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\rho^{-1} \eta \Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}) \right) \right| dx dt \\
& + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot [\rho^{-1} \eta (\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T \right. \right. \\
& \quad \left. \left. \cdot (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\ell_1+\ell_2=\ell} C_\beta^{\ell_1} \Gamma^{\ell_2} \eta \partial_t^2 \Gamma^{\ell_1} \mathbf{v})] \right) \right| dx dt
\end{aligned} \tag{3.21}$$

For the first term on the right-hand side of (3.21), we have

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\rho^{-1} \eta \Delta \Gamma^\alpha \mathbf{v}) \right) \right| dx dt \\
& \lesssim \sum_{i=1,2,3} \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot \partial_i (\rho^{-1} \eta \partial_i \Gamma^\alpha \mathbf{v}) \right) dx dt \\
& \quad + \sum_{i=1,2,3} \int_0^T \int_{\mathbb{R}^3} f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot [\partial_i (\rho^{-1} \eta) \partial_i \Gamma^\alpha \mathbf{v}] \right) dx dt \\
& \lesssim \int_0^T \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^\infty} \left(\|\langle r \rangle \eta\|_{L^\infty} + \|\langle r \rangle \nabla (\rho^{-1} \eta)\|_{L^3} \right) \\
& \quad \cdot \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} |\nabla \Gamma^\alpha \mathbf{v}| \right\|_{L^2} dt \\
& \lesssim \delta L \mathcal{E}_k(T).
\end{aligned}$$

The second term on the right-hand side of (3.21) can be solved as (3.19). We employ the analogous method utilized for the first two terms on the right-hand side of (3.4) to solve the last two terms on the right-hand side of (3.21).

For the second term on the right-hand side of (3.20), we have

$$\begin{aligned}
& \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \neq |\alpha|}} \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla^\perp \cdot \left(\nabla \eta \Delta^{-1} \nabla^\perp \cdot (\Gamma^\gamma \eta \partial_t^2 \Gamma^\beta \mathbf{v}) \right) \right| dx dt \\
& \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^\infty} \|\langle r \rangle \Gamma^\gamma \eta\|_{L^3} \|\langle t-r \rangle \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^2} dt \\
& \quad + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^\infty} \|\langle r \rangle \Gamma^\gamma \eta\|_{L^2} \|\langle t-r \rangle \partial_t^2 \Gamma^\beta \mathbf{v}\|_{L^3} dt \\
& \lesssim \delta L \mathcal{E}_k(T) + \delta \int_0^T \langle t \rangle^{-2} \mathcal{E}_k(t) dt.
\end{aligned}$$

For the last term on the right-hand side of (3.17), by Lemmas 2 and 6, we have

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} \left| f_k \left(\partial_r \Gamma^\alpha \mathbf{v} + \frac{\Gamma^\alpha \mathbf{v}}{r} \right) \cdot \nabla \Delta^{-1} \nabla_i^\perp \left(\nabla_i \eta \Delta^{-1} \nabla^\perp \cdot [C_\alpha^\beta (\nabla X)^{-T} (\nabla \Gamma^\gamma \mathbf{v})^T \right. \right. \\
& \quad \left. \left. \cdot (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1 + \iota_2 = \beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}) \right] \right) \Big| dx dt \\
& \lesssim \sum_{\substack{\beta + \gamma = \alpha \\ |\gamma| < |\beta| < |\alpha|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^\infty} \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^3} \\
& \quad \cdot \left\| \langle t \rangle (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1 + \iota_2 = \beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}) \right\|_{L^2} dt \\
& \quad + \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \leq |\gamma|}} \int_0^T \langle t \rangle^{-1} \left\| r^{-\frac{1}{4}} \langle r \rangle^{-\frac{1}{2}} \left(|\partial_r \Gamma^\alpha \mathbf{v}| + \frac{|\Gamma^\alpha \mathbf{v}|}{r} \right) \right\|_{L^2} \|\langle r \rangle \nabla \eta\|_{L^\infty} \|\nabla \Gamma^\gamma \mathbf{v}\|_{L^2} \\
& \quad \cdot \left\| \langle t \rangle (\partial_t^2 \Gamma^\beta \mathbf{v} - \Delta \Gamma^\beta \mathbf{v} + \sum_{\iota_1 + \iota_2 = \beta} C_\beta^{\iota_1} \Gamma^{\iota_2} \eta \partial_t^2 \Gamma^{\iota_1} \mathbf{v}) \right\|_{L^3} dt \\
& \lesssim \delta L \mathcal{E}_\kappa(T) + \int_0^T \langle t \rangle^{-2} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt.
\end{aligned}$$

Combining all the estimates, we conclude that

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\partial \Gamma^\alpha \mathbf{v}|^2(t) dx + L \mathcal{E}_\kappa(T) \\
& \leq C_0 \int_{\mathbb{R}^3} |\partial \Gamma^\alpha \mathbf{v}|^2(0) dx + C C_0 (\delta + \sup_{0 \leq t \leq T} \mathcal{E}_\kappa^{\frac{1}{2}}(t)) L \mathcal{E}_\kappa(T) + C \int_0^T \langle t \rangle^{-\frac{4}{3}} \mathcal{E}_\kappa^{\frac{3}{2}}(t) dt + C \delta \int_0^T \langle t \rangle^{-\frac{4}{3}} \mathcal{E}_\kappa(t) dt,
\end{aligned}$$

where $C > 0$ is some positive constant. By the smallness of δ , $\mathcal{E}_\kappa(t)$, and the standard continuity method, we arrive at the main result.

Author contributions

All authors contributed equally.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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