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Review

Moving surfaces and interfaces : application to damage, fracture and wear contact

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Abstract: The full scenario of the degradation of solids under mechanical loading is described by modelling the gradual loss of rigidity. This common approach is purely local. Another way to describe the damage evolution is to consider the propagation of the surface separating sound material and damaged material. When this surface is moving, a flux of matter is induced, that is useful for describing the loss of material during wear mechanisms or brittle fracture. The article proposes modelling of moving surface and interface in order to describe such behaviours. The problem of evolution is written, analysis of stability and bifurcation of the propagation is also presented. Applications to brittle fracture, transition from fracture to damage and wear contact are briefly investigated.

Keywords: moving interface; damage; wear contact; propagation of discontinuities

1. Introduction

Two main families of methods exist to model failure of quasi-brittle structures. One is based on crack models, another one uses continuous damage approach that leads to a local loss of stiffness. Initiation of crack can be investigated using Griffith law an a criterion of global stability as proposed in [1], but fracture mechanics is generally not sufficient to model the full scenario of the degradation of solids. Another point of view is adopted here based on damage modelling.

The full scenario of the degradation of solids under mechanical loading is described by modelling the gradual loss of rigidity. This common approach is purely local. Generally local damage induces localization. Several models were proposed to avoid spurious localization: non local approaches and higher order, damage based, gradient models. These approaches are compared in [2]. Other new approaches are also proposed, the phase field approach [3] and the so-called variational approach [4].

Another way to describe the damage evolution is to consider the propagation of the surface separating the sound material and damaged material, as proposed for brittle material in [5, 6]. This point of view is adopted here.

The propagation of surface inside a body is analysed. The moving surface here is associated with a change of mechanical properties. This framework is used to describe damage or phase transformation. Variational formulations were performed to describe the evolution of the surface between undamaged and damaged material. Connection with the notion of configurational forces can be made [7]. When this surface is moving, a flux of matter is induced, that is useful for describing the loss of material during wear mechanisms or brittle fracture [8].

For elastic brittle material the evolution of the interface separating the undamaged material ($d = 0$) from the total damaged ($d = 1$) have been studied using an energetic description of the propagation of damage [9, 10]. In this description the damage parameter jumps from 0 to 1.

The interface has no thickness and mechanical quantities present strong discontinuities. The evolution in terms of rate of displacement and velocity of propagation of the surface is governed by a variational inequality. Criteria of stability and uniqueness have been established. For example, nucleation of defects in this modelling can be considered as a bifurcation of equilibrium solution [11]. Description of moving interfaces and of moving layers is also a manner for studying loss of material. Such a description of thin or thick layers permits to describe complex processes of wear contact between two bodies in relative motion [8, 12]. In a more recent paper [13], the transition between undamaged material to damaged material is continuous through a layer of finite thickness. The evolution of damage is then associated to a moving layer.

Two cases have been considered: the propagation of a interface and the propagation of a moving layer of finite thickness.

For sharp interface, the transition zone is very thin, the damage parameter has a discontinuity, it jumps from 0 to 1. The driving force associated to the propagation of the interface is a local release rate of energy $G(s)$ as in [14]. Moreover, stability and non bifurcation conditions of the evolution are given when a normality rule based on $G(s)$ governs the propagation of the interface. When an additional surface energy is considered along the interface, the propagation is more stable [15].

For a moving layer, the transition is more regular. The thickness of the transition provides a length scale in the model. For specific definition of the damage, a local condition of steady state is induced and a generalisation of the local energy release rate is obtained. This is the driving force associated to the motion of surface separating the sound material from the damage one.

The main ideas of the model is to consider that the strain energy changes from a sound material to a damaged one depending on a damage parameter, which is continuous or not, and to describe the motion of the interface by a complementary law. For particular description, the motion of the interface induces a dissipation depending on an energy release rate. A normality rule is then proposed to govern the propagation of the interface based on this mechanical quantities. This choice generalizes classical Griffith's law.

The purpose of this article is to describe the motion of a surface or interface, to propose an evolution law for this motion in terms of the local energy release rate. The rate boundary value problem is then analysed and conditions of stability and bifurcation are given. Some applications and comments of this modelling are proposed in the last section.

2. General Preliminaries

Consider a body Ω. The external boundary $\partial\Omega$ is decomposed in two complementary parts: $\partial\Omega_u$ where the displacement $\underline{u}$ is prescribed $\underline{u}^d(t)$ and $\partial\Omega_T$ where the loading $\underline{T}^d(t)$ is applied: $\partial\Omega = \partial\Omega_u \cup \partial\Omega_T$, $\quad \emptyset = \partial\Omega_u \cap \partial\Omega_T$. The displacement $\underline{u}$ is continuous across the surface Γ_o, separating the undamaged material and the damaged material. The internal state of stress $\boldsymbol{\sigma}$ is such that the stress vector is also continuous along the surface Γ_o. Then we have the relations of continuity:

$$[\underline{u}]_\Gamma = 0, \quad [\boldsymbol{\sigma}]_\Gamma.\underline{\nu} = 0, \tag{1}$$

where $\underline{\nu}$ is the normal vector to the interface. When the surface is moving, these conditions must be preserved.

2.1. The geometry of a surface

Consider the surface Γ_o separating the sound material from the damaged material. The surface is parametrized by two parameters s^α. For $\mathbf{M}_o$ on the surface, the tangent space is determined by the local basis $\underline{T}_\alpha$. The normal vector to the tangent space is $\underline{\nu}$. Then we have

$$\underline{T}_\alpha = \frac{\partial \mathbf{M}_o}{\partial s^\alpha} = \nabla_\alpha \mathbf{M}_o, \quad \nabla_\alpha \underline{\nu} = -\boldsymbol{K}.\underline{T}_\alpha, \tag{2}$$

where $\boldsymbol{K}$ is the curvature tensor at point $\mathbf{M}_o$.

When this surface is moving the local basis changes accordingly. At time t the point $\mathbf{M}_o^t$ belongs to the surface, it satisfies the equation $\phi(\mathbf{M}_o^t, t) = 0$ defining Γ_o.

At time $t^+ = t + \mathrm{d}t$, the geometrical point $\mathbf{M}_o^t$ comes in $\mathbf{M}_o^{t^+}$ such that $\phi(\mathbf{M}_o^{t^+}, t^+) = 0$ then the new position is $\mathbf{M}_o^{t^+} = M_o^t - \mathrm{d}t\, a(s^\alpha)\, \underline{\nu}_t$ where the normal speed a satisfies the relation

$$\frac{\partial \phi}{\partial t} - a \nabla\phi.\underline{\nu}_t = 0. \tag{3}$$

Any quantities f defined at $\mathbf{M}_o$ is varying during the motion. Introducing the derivative of f following the motion of Γ_o by

$$D_a f = \lim_{\Delta t \to 0} \frac{f(\mathbf{M}_o^{t^+}, t^+) - f(\mathbf{M}_o^t, t)}{\mathrm{d}t}. \tag{4}$$

We obtain for the evolution of the geometry

$$D_a \underline{T}_\alpha = -\nabla_\alpha a\, \underline{\nu}, \quad D_a \underline{\nu} = \nabla a \quad D_a \boldsymbol{K} = -\nabla\nabla a - \boldsymbol{K}.\boldsymbol{K} a. \tag{5}$$

When f is the value at $\mathbf{M}_o$ of a field defined on Ω, then the derivative $D_a f$ is the classical convective derivative

$$D_a f = \frac{\partial f}{\partial t} - a \nabla f.\underline{\nu}. \tag{6}$$

This expression is useful for written conservation laws and continuities along the surface Γ_o.

Hadamard's relations. On account of perfect bounding between phases, displacement and stress vectors are continuous along Γ_o. Their rates have discontinuities according to the general compatibility conditions of Hadamard. These conditions are rewritten in terms of convected derivative:

$$0 = [\underline{u}]_\Gamma, \qquad D_a([\underline{u}]_\Gamma) = [\underline{v}]_\Gamma - a\,[\nabla\underline{u}]_\Gamma.\underline{\nu} = 0, \tag{7}$$

$$0 = [\boldsymbol{\sigma}]_\Gamma.\underline{\nu}, \quad D_a([\boldsymbol{\sigma}]_\Gamma.\underline{\nu}) = [\dot{\boldsymbol{\sigma}}]_\Gamma.\underline{\nu} + \mathrm{div}_{\Gamma_o}[a\boldsymbol{\sigma}]_\Gamma = 0. \tag{8}$$

The last equation is obtained by taking account of the conservation of the momentum. Indeed we have

$$D_a([\boldsymbol{\sigma}]_\Gamma.\underline{\nu}) = D_a[\boldsymbol{\sigma}]_\Gamma.\underline{\nu} + \boldsymbol{\sigma}.D_a(\underline{\nu}) = 0, \tag{9}$$

using the conservation of the momentum in the local frame relatively to Γ_o

$$\underline{T}_\alpha.\nabla.\boldsymbol{\sigma}.\underline{T}_\alpha + \underline{\nu}.\nabla\boldsymbol{\sigma}.\underline{\nu} = 0, \tag{10}$$

and the expression of the surface divergence

$$\mathrm{div}_{\Gamma_o} f = \mathrm{div}\, f - \underline{\nu}.\nabla f.\underline{\nu}, \tag{11}$$

the required result is obtained.

In two dimensions. In two dimensions the expressions are:

$$D_a\underline{T} = -\frac{\mathrm{d}a}{\mathrm{d}S}\,\underline{\nu}, \quad D_a\underline{\nu} = \frac{\mathrm{d}a}{\mathrm{d}S}\,\underline{T}, \quad D_aK = -\frac{\mathrm{d}^2a}{\mathrm{d}S^2} - K^2\,a, \tag{12}$$

$$0 = [\underline{v}]_\Gamma - a\nabla\underline{u}.\underline{\nu}, \quad 0 = [\dot{\boldsymbol{\sigma}}]_\Gamma.\underline{\nu} + \frac{\mathrm{d}}{\mathrm{d}S}([\boldsymbol{\sigma}.\underline{T}]_\Gamma a). \tag{13}$$

3. The Sharp Interface

The domain is composed of two distinct volumes Ω_o and Ω_1, which are occupied by two materials with different mechanical characteristics. The perfect interface between them is assumed to be a regular surface and is denoted by Γ_o. Material 0 changes into material 1, along Γ_o by an irreversible process. Hence, Γ_o moves with a normal velocity a, the irreversibility is given by the positivity of a for a propagation in the direction external to 2.

The subscript i is used to denote material i. The actual state is characterized by the displacement field $\underline{u}$. The strain field $\boldsymbol{\varepsilon}$ is given by $\boldsymbol{\varepsilon}(\underline{u}) = \frac{1}{2}(\nabla\underline{u} + \nabla^T\underline{u})$.

To simplify the presentation, we consider here that the materials are linear elastic. The total potential energy of the structure has the following form:

$$\mathcal{E}(\underline{u}, \Gamma_o) = \sum_i \int_{\Omega_i} \psi_i(\boldsymbol{\varepsilon}(\underline{u}))\,\mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{T}^d.\underline{u}\,\mathrm{d}S, \tag{14}$$

where $\psi_i(\boldsymbol{\varepsilon})$ denotes the density of strain energy in the domain i: $\psi_i(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}_i : \boldsymbol{\varepsilon}$.

It is important to point out that the potential energy represents the global free energy in a thermodynamic description; then the position of the interface is an internal parameter for the global structure.

The displacement $\underline{u}$ is kinematically admissible, it satisfies particular boundary condition:

$$\underline{u}(x,t) \in C = \left\{ \underline{u} / \underline{u}(x,t) = \underline{u}^d(t), x \in \partial\Omega_u, [\underline{u}]_\Gamma = 0, x \in \Gamma_o(t) \right\}. \tag{15}$$

At any time, when the position of the interface is known, the behaviour of the body is those of a composite bi-materials, with perfect contact between two material phases. Then the solution of the problem of equilibrium is determined by the minimization of the total potential energy with respect to the displacement $\underline{u}$ among the set of admissible fields C.

Characteristic of an equilibrium state. For a state of equilibrium ($a = 0$) the variations with respect to $\underline{u}$ are

$$\frac{\partial \mathcal{E}}{\partial \underline{u}}.\delta \underline{u} = \sum_i \int_{\Omega_i} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \underline{u}) \, \mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{T}^d.\delta \underline{u} \, \mathrm{d}\Omega. \tag{16}$$

A state of equilibrium is defined by displacement $\underline{u}$ such that

$$\boldsymbol{\sigma} = \mathbb{C}_i : \boldsymbol{\varepsilon}(\underline{u}), \text{ in } \Omega_i, \quad \operatorname{div} \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma}.\underline{n} = \underline{T}^d \text{ along } \partial\Omega_T. \tag{17}$$

Variations with respect to the position of the interface. The total variation of $\mathcal{E}$ near a position of equilibrium, must take into account of the continuity condition. The variations of the displacement and the variations of the position of the surface are coupled by Hadamard's relation.

$$[\delta \underline{u}]_\Gamma + \delta a [\nabla \underline{u}]_\Gamma.\underline{\nu} = 0. \tag{18}$$

The total variation of the potential energy is:

$$\frac{\partial \mathcal{E}}{\partial \underline{u}}.\delta \underline{u} + \frac{\partial \mathcal{E}}{\partial \Gamma_o}.\delta \Gamma_o = \int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\delta \underline{u}) \, \mathrm{d}\Omega - \int_{\Gamma_o} [\psi]_\Gamma \delta a \, \mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{T}^d.\delta \underline{u} \, \mathrm{d}S. \tag{19}$$

Taking account of the equilibrium, the total variation of $\mathcal{E}$ near an equilibrium point is reduced to

$$-\frac{\partial \mathcal{E}}{\partial \Gamma_o}.\delta \Gamma_o = \int_{\Gamma_o} \delta a \Big([\psi]_\Gamma - \underline{\nu}.\boldsymbol{\sigma}.[\nabla \underline{u}]_\Gamma.\underline{\nu} \Big) \, \mathrm{d}S = \int_{\Gamma_o} \mathcal{G}(\mathbf{M}_o) \delta a(\mathbf{M}_o) \, \mathrm{d}S = D_m. \tag{20}$$

This corresponds to the dissipation D_m of the system as pointed out in [10, 12]. We recover the results of [7, 14, 16]. The definition of the release rate of energy can be written in a simpler form, taking account of continuity of the stress vector and of the fact the discontinuities of strain and stress are orthogonal [17]:

$$[\boldsymbol{\sigma}]_\Gamma.[\nabla \underline{u}]_\Gamma = 0, \tag{21}$$

then the release rate of energy $\mathcal{G}$ takes the final expression:

$$\mathcal{G} = [\psi]_\Gamma - \boldsymbol{\sigma} : [\nabla \underline{u}]_\Gamma. \tag{22}$$

The release rate of energy depends upon the position of the point M_o^t and on the loading.

3.1. The quasi-static evolution

From a position of equilibrium, we applied an amount of loading. The body is deformed an simultaneously, the surface Γ_o is moving accordingly some constitutive law. A kinetic law defined by a direct relation between a and $\mathcal{G}$ can be chosen as proposed in [16]. To describe the irreversibility of the motion, we specify here a different relation.

The propagation law. Based on the form of the dissipation, a criterion on $\mathcal{G}$ is chosen as a generalized form of the well-known theory of Griffith. We assume a normality rule to govern the speed of propagation a

$$a \geq 0, \quad \mathcal{G} \leq G_c, \quad (\mathcal{G} - G_c)a = 0. \tag{23}$$

The subset of Γ_o where the critical value G_c is reached is denoted Γ_o^+. Then the propagation is only possible on point of Γ_o^+ where $\mathcal{G}(M_o^t, t) = G_c$, this is equation of a surface.

When the interface is moving, the consistency condition associated to the motion is given by the convected derivative of $\mathcal{G}$: during the motion $D_a\mathcal{G} = 0$. This leads to the consistency condition written for all points belonging to Γ_o^+:

$$(a - a^*)\; D_a\mathcal{G} \; \geq 0, \quad \forall a^* \geq 0 \text{ on } \Gamma_o^+. \tag{24}$$

Evaluation of $D_a\mathcal{G}$. To calculate the convected derivative, we derive term by term, using the Hadamard's relations on velocities

$$\underline{\mathrm{v}}_o + a\nabla\underline{u}_o.\underline{\nu} = \underline{\mathrm{v}}_1 + a\nabla\underline{u}_1.\underline{\nu}. \tag{25}$$

The first term of $\mathcal{G}$ is the jump of energy

$$D_a[\psi]_\Gamma = -\boldsymbol{\sigma}_1 : (\nabla\underline{\mathrm{v}}_1 + a\nabla\nabla\underline{u}_1.\underline{\nu}) + \boldsymbol{\sigma}_o : (\nabla\underline{\mathrm{v}}_o + a\nabla\nabla\underline{u}_o.\underline{\nu}). \tag{26}$$

Then we get

$$D_a\mathcal{G} = [\psi]_\Gamma - D_a\boldsymbol{\sigma}_1 : [\nabla\underline{u}]_\Gamma - \boldsymbol{\sigma}_1 : [D_a\nabla\underline{u}]_\Gamma = [\boldsymbol{\sigma}]_\Gamma : \nabla\underline{\mathrm{v}}_o - \dot{\boldsymbol{\sigma}}_1 : [\nabla\underline{u}]_\Gamma - aG_n, \tag{27}$$

where $G_n = -[\boldsymbol{\sigma}]_\Gamma : \nabla\nabla\underline{u}_1.\underline{\nu} + \nabla\boldsymbol{\sigma}_1.\underline{\nu} : [\nabla\underline{u}]_\Gamma$.

The rate boundary value problem. A solution $(\underline{\mathrm{v}}, a)$ of the problem of evolution must satisfy

- the constitutive law :

$$\dot{\boldsymbol{\sigma}} = \mathbb{C}_i : \boldsymbol{\varepsilon}(\underline{\mathrm{v}}) \text{ in } \Omega_i, \tag{28}$$

- the compatibility of the velocity :

$$\boldsymbol{\varepsilon}(\underline{\mathrm{v}}) = \frac{1}{2}(\nabla\underline{\mathrm{v}} + \nabla^t\underline{\mathrm{v}}), \tag{29}$$

and the boundary condition $\underline{\mathrm{v}} = \underline{\mathrm{v}}^d$ on $\partial\Omega_u$,

- the conservation of the momentum:

$$\operatorname{div} \dot{\boldsymbol{\sigma}} = 0, \text{ in } \Omega, \tag{30}$$

and $\dot{\boldsymbol{\sigma}}.\underline{n} = \underline{\dot{T}}^d$ on $\partial\Omega_T$,

- the compatibility conditions on the moving interface:

$$[D_a \underline{u}]_\Gamma = 0, \quad [D_a(\boldsymbol{\sigma}.\underline{\nu})]_\Gamma = 0, \tag{31}$$

- the consistency condition:

$$\forall a^* \geq 0 \text{ on } \Gamma_o^+, \quad (a - a^*) D_a \mathcal{G} \geq 0. \tag{32}$$

Theorem 1. *The evolution is determined by the variational inequality*

$$\frac{\partial \mathcal{F}}{\partial \underline{\mathrm{v}}}.(\underline{\mathrm{v}}^* - \underline{\mathrm{v}}) + \frac{\partial \mathcal{F}}{\partial a}(a^* - a) \geq 0, \tag{33}$$

among the set K.A of admissible fields $(\underline{\mathrm{v}}^*, a^*)$*:*

$$\begin{aligned} K.A &= \left\{ (\underline{\mathrm{v}}, a) | \underline{\mathrm{v}} = \underline{\mathrm{v}}^d \text{on } \partial\Omega_u, [\underline{\mathrm{v}}]_\Gamma + a[\nabla \underline{u}]_\Gamma = 0, a \in K \right\}, \\ K &= \{ a | a \geq 0 \text{ on } \Gamma_o^+, a = 0 \text{ otherwise } \}, \end{aligned}$$

and the functional $\mathcal{F}(\underline{\mathrm{v}}, a, \underline{\dot{T}}^d)$ *is defined as*

$$\mathcal{F} = \sum_i \int_{\Omega_i} \frac{1}{2} \boldsymbol{\varepsilon}(\underline{\mathrm{v}}) : \mathbb{C}_i : \boldsymbol{\varepsilon}(\underline{\mathrm{v}}) \, \mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{\dot{T}}.\underline{\mathrm{v}} \, \mathrm{d}S - \int_{\Gamma_o} a[\boldsymbol{\sigma}]_\Gamma : \nabla \underline{\mathrm{v}}_1 \, \mathrm{d}S + \int_{\Gamma_o} \frac{1}{2} a^2 G_n \, \mathrm{d}S. \tag{34}$$

The equations (28,29,30) are those of a classical problem of elasticity with non classical boundary conditions (31) on Γ_o^+. Assume that a is a given function on Γ_o^+ then the problem can be solved with respect to $\underline{\mathrm{v}}$. This solution $\underline{\mathrm{v}}$ is a function of the boundary conditions and of the distribution of the speed a along the interface. We can define $\mathcal{W}(a)$ the value of $\mathcal{F}$ for this solution then:

$$\mathcal{W}(a) = \mathcal{F}(\underline{\mathrm{v}}(a, \underline{\dot{T}}^d, \underline{u}^d), a, \underline{\dot{T}}^d), \tag{35}$$

and finally the problem of evolution is defined as a variational inequality based on $\mathcal{W}$. It is obvious that $\mathcal{W}$ is quadratic in a and we have the properties

Stability If $a.\dfrac{\partial^2 \mathcal{W}}{\partial a \partial a}.a \geq 0$ for all $a \geq 0$ over Γ_o^+, the position $a = 0$ is stable.

No- bifurcation If $a.\dfrac{\partial^2 \mathcal{W}}{\partial a \partial a}.a \geq 0$ for all a over Γ_o^+, there is no bifurcation.

When an additional surface energy is present on the interface Γ_o^+ the release rate of energy is changed for a motion of the surface

$$\delta \int_{\Gamma_o} \beta \, \mathrm{d}S = \int_{\Gamma_o} \beta \operatorname{Tr} \boldsymbol{K} \delta a \, \mathrm{d}S, \quad \mathcal{G}_\beta = \mathcal{G}_o - \beta \operatorname{Tr} \boldsymbol{K}. \tag{36}$$

4. A Continuous Transition

Now, we consider that the interface has a finite thickness. The damage parameter varies from 0 to 1 continuously. The material of the body has an elastic linear behaviour with moduli evolving with damage. Then the local free energy $\psi(\boldsymbol{\varepsilon}, \mathrm{d})$ is a function of the strain and of the damage parameter d. The state equations are defined classically as:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \mathbb{C}(\mathrm{d}) : \boldsymbol{\varepsilon}, \quad Y = -\frac{\partial \psi}{\partial \mathrm{d}}. \tag{37}$$

The elastic moduli are known functions of d. Driving force Y is associated to the damage parameter and the dissipation of the whole system is reduced to

$$D_m = \int_\Omega Y \dot{\mathrm{d}} \, \mathrm{d}\Omega. \tag{38}$$

When damage is established, the whole body is decomposed in three parts: the undamaged body Ω_o where d = 0, the transition zone Ω_c and the damaged material Ω_1 where d = 1.

On the boundary $\partial\Omega_c \cap \partial\Omega_o = \Gamma_o$ the moduli of elasticity are continuous, the displacement and the stress vector being continuous, the strain is also continuous then the free energy is continuous. Therefore, when the surface Γ_o is moving, there is no dissipation along the boundary of this layer.

The level-set $\phi = 0$ gives the position of Γ_o. We assume that the damage d is a continuous explicit function $\mathrm{d}(\phi)$ of the distance ϕ to the surface Γ_o. In the domain Ω the damage parameter satisfies

$$0 \leq \mathrm{d}(\phi) \leq 1, \quad \begin{cases} \mathrm{d} = 0, & \phi \leq 0, \\ d'(\phi) \geq 0, & 0 \leq \phi \leq l_c, \\ \mathrm{d}(\phi) = 1, & l_c \leq \phi. \end{cases} \tag{39}$$

The surface iso-damage $\mathrm{d}(\mathbf{M}, t) = d_o$ is a level-set, it corresponds to the level-set $\phi(\mathbf{M}, t) = z$. The evolution of this level set is those of the motion of a surface then

$$\dot{\phi} + a \nabla\phi . \underline{\nu} = 0, \tag{40}$$

as ϕ is a function distance $\underline{\nu} = \nabla\phi$, $||\underline{\nu}|| = 1$ and $\dot{\phi} + a = 0$ and we have

$$\dot{\mathrm{d}} + a \nabla \mathrm{d} . \underline{\nu} = \dot{\mathrm{d}} + a \, d'(\phi) = 0, \quad d'(\phi) = \nabla \mathrm{d} . \underline{\nu}. \tag{41}$$

4.1. On geometry of the layer

A point of the layer is described by the parameters s^α along Γ_o and the distance to Γ_o.

$$\mathbf{M} = \mathbf{M}_o + z\underline{\nu}. \tag{42}$$

At point $\mathbf{M}$ the local basis is $(\underline{t}_\alpha, \underline{\nu})$ with

$$\underline{t}_\alpha = \underline{T}_\alpha - zK^\beta_\alpha \, \underline{T}_\beta. \tag{43}$$

Reciprocal basis is $\underline{t}^\alpha$ such that $\underline{t}^\alpha . \underline{t}_\beta = \delta^\alpha_\beta$. To simplify the analysis we consider now that the tangent

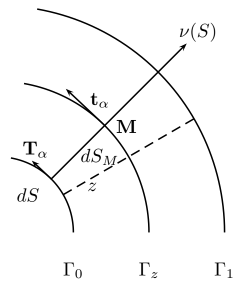

Figure 1. The geometry of the layer, the unit area at point M depends on the curvature of Γ_o.

vectors $\underline{T}_\alpha$ are eigenvectors of the curvature $\boldsymbol{K}$, then

$$\underline{t}_1 = (1 - zK_1)\underline{T}_1, \quad \underline{t}_2 = (1 - zK_2)\underline{T}_2. \tag{44}$$

The metric tensor in this basis is then

$$g_{11} = (1 - zK_1)^2, \quad g_{22} = (1 - zK_2)^2, \quad g_{12} = 0. \tag{45}$$

And the curvature at point **M** is determined by

$$\nabla\underline{\nu} = \frac{\partial\underline{\nu}}{\partial s^\alpha} \otimes \underline{t}^\alpha = -K^\beta_\alpha \underline{T}_\beta \otimes \underline{t}^\alpha = -\sum_\alpha \frac{K_\alpha}{1 - zK_\alpha}\underline{T}_\alpha \otimes \underline{T}_\alpha = -\kappa_\alpha \underline{t}_\alpha \otimes \underline{t}^\alpha. \tag{46}$$

We recover the classical result: $\mathrm{Tr}\,\nabla\underline{\nu} = \Delta_S\phi = -\mathrm{Tr}\,\boldsymbol{\kappa}$. We define the covariant derivative of any quantity f by

$$\nabla_S f = \frac{\partial f}{\partial s^\alpha}\underline{t}^\alpha = g^{\alpha\beta}\frac{\partial f}{\partial s^\alpha}\underline{t}_\beta = \nabla_\alpha f\ \underline{t}^\alpha, \tag{47}$$

then we obtain:

$$\nabla_S\nabla_S f = \nabla_\gamma(g^{\alpha\beta}\frac{\partial f}{\partial s^\alpha}\ \underline{t}_\beta) \otimes \underline{t}^\gamma, \tag{48}$$

$$\Delta_S f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial s^\alpha}(g^{\alpha\beta}\sqrt{g}\frac{\partial f}{\partial s^\beta}), \tag{49}$$

$$g = det(\boldsymbol{g}), \quad \Gamma^\alpha_{\alpha\beta} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial s^\alpha}, \tag{50}$$

$$\mathrm{d}\Omega = \sqrt{g}\mathrm{d}s^1\mathrm{d}s^2\mathrm{d}z. \tag{51}$$

Where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbol associated to the tangent frame :

$$\frac{\partial\underline{t}_\alpha}{\partial s^\beta} = \Gamma^\gamma_{\alpha\beta}\underline{t}_\gamma + \kappa_{\alpha\beta}\underline{\nu}. \tag{52}$$

When the surface Γ_o is moving with normal velocity $a(s^\alpha)$ the local basis is changing accordingly and in particular

$$D_a\underline{\nu} = -\nabla a, \quad D_a\boldsymbol{\kappa} = \boldsymbol{\kappa}.\boldsymbol{\kappa} \quad a + \nabla_S\nabla_S a, \quad D_a(\ \mathrm{d}\Omega) = D_a(\sqrt{g})\ \mathrm{d}\Omega. \tag{53}$$

In the same spirit $\phi(M_o+z\underline{v}, t) = z$. As $D_a(M_o+z\underline{v}) = -a\underline{v}+zD_a\underline{v}$ the velocity of the level-set $\phi(M, t) = z$ for the point $M = M_o + z\underline{v}$ is the same than at the point M_o. All point on the normal direction to Γ_o has the same normal velocity a. Then the damage parameter d satisfies $\dot{d} + a\nabla \mathrm{d}.\underline{v} = 0$. The profile of damage d is conserved along the normal direction $d' = \nabla \mathrm{d}.\underline{v}$.

4.2. The position of equilibrium

The total potential energy is defined as previously

$$\mathcal{E}(\underline{u}, \mathrm{d}, \underline{T}^d) = \int_\Omega \psi(\boldsymbol{\varepsilon}(\underline{u}), \mathrm{d})\, \mathrm{d}\Omega - \int_{\partial\Omega_T} \underline{T}^d.\underline{u}\, \mathrm{d}S. \tag{54}$$

The equilibrium position minimises the potential energy among the set of admissible displacement $\underline{u}$

$$\underline{u} \in C = \left\{\underline{u} | \underline{u} = \underline{u}^d, \text{ over } \partial\Omega_u\right\}, \tag{55}$$

then

$$\boldsymbol{\sigma} = \mathbb{C}(\mathrm{d}) : \boldsymbol{\varepsilon}(\underline{u}), \quad 0 = \operatorname{div} \boldsymbol{\sigma}, \quad \boldsymbol{\sigma}.\underline{n} = \underline{T} \text{ over } \partial\Omega_T. \tag{56}$$

Combining the equilibrium position and the total variations of $\mathcal{E}$, we obtain

$$-\frac{\partial \mathcal{E}}{\partial \mathrm{d}}.\delta \mathrm{d} = \int_\Omega Y \delta \mathrm{d}\, \mathrm{d}\Omega = D_m, \tag{57}$$

which correspond to the mechanical dissipation of the system.

4.3. Dissipation of the system

The dissipation associated to the motion of the layer is associated to the motion of the surface Γ_o

$$D_m = \int_\Omega Y \dot{\mathrm{d}}\, \mathrm{d}\Omega = \int_\Omega Y \mathrm{d}' \dot{\phi}\, \mathrm{d}\Omega = \int_{\Gamma_o} (\int_o^l Y \mathrm{d}' j(z) dz) a(M_o)\, \mathrm{d}S. \tag{58}$$

This relation defines a generalized driving force associated to the motion of the layer ; the local thickness l depends on the point $\mathbf{M}_o$ and is limited by l_c:

$$\mathcal{G}(M_o) = \int_o^l Y \mathrm{d}' j(z) \mathrm{d}z. \tag{59}$$

The volume is described by the geometry of the surface Γ_o and the distance z to this surface. The tensor of curvature at M_o is $\boldsymbol{K}$ and the curvature at $M = M_o + z\underline{v}$ depends on $\boldsymbol{K}$ and z. The unit of area defined on the surface $\phi = z$ is related to the unit of area defined on $\phi = 0$ as

$$\mathrm{d}S_M = det(\boldsymbol{I} - z\boldsymbol{K})\, \mathrm{d}S_o. \tag{60}$$

Choice of a propagation law. The velocity a is determined with respect to an evolution law based on the driving force $\mathcal{G}$. The integration of the local normality rule:

$$\dot{\mathrm{d}} \geq 0, \quad Y \leq Y_c, \quad (Y - Y_c)\dot{\mathrm{d}} = 0, \tag{61}$$

suggests that the velocity a satisfies the generalized normality rule:

$$a(M_o) \geq 0, \quad \mathcal{G}(M_o) \leq G_c(M_o) = \int_o^l Y_c \nabla \mathrm{d}.\underline{v} j(z) \mathrm{d}z; \quad \Big(\mathcal{G}(M_o) - G_c(M_o)\Big) a(M_o) = 0. \tag{62}$$

4.4. *The evolution of the layer*

Now the problem of evolution is investigated. During the motion of the layer, the critical value on $\mathcal{G}$ is conserved. We are interested by the evolution of integral of a continuous function f like

$$F = \int_{\Omega_c} f \, \mathrm{d}\Omega \rightarrow \frac{\mathrm{d}F}{\mathrm{d}t} = \int_{\Omega_c} (D_a f + f \,\mathrm{Tr}\, \nabla D_a M) \, \mathrm{d}\Omega. \tag{63}$$

As $M = M_o + z\underline{\nu}$, we obtain $D_a M = a\underline{\nu} - z\nabla a$ and $\mathrm{Tr}\, \nabla D_a \mathbf{M} = a \,\mathrm{Tr}\, \nabla \underline{\nu} - z\Delta a$.

Consider the particular function $f(\boldsymbol{\varepsilon}, \mathrm{d}) = (Y - Y_c)\mathrm{d}'$ and a virtual velocity fields a^*, then the normality law implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_c} f a^* \, \mathrm{d}\Omega = \int_{\Omega_c} (D_a f + f \,\mathrm{Tr}\, \nabla D_a \mathbf{M}\,) a^* \, \mathrm{d}\Omega. \tag{64}$$

Due to the hypothesis $\mathrm{d}(\phi)$ then $D_a \mathrm{d} = 0$ and

$$D_a f = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : D_a \boldsymbol{\varepsilon} = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : (\dot{\boldsymbol{\varepsilon}} + \nabla \boldsymbol{\varepsilon}. D_a \mathbf{M}) = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : (\dot{\boldsymbol{\varepsilon}} + a \nabla \boldsymbol{\varepsilon}. \underline{\nu}) - z \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : (\nabla \boldsymbol{\varepsilon}. \nabla a). \tag{65}$$

The last term is combined with $f \,\mathrm{Tr}\, \nabla D_a \mathbf{M}$ and taking account of $\nabla a.\underline{\nu} = 0$

$$\Big(z \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : (\nabla \boldsymbol{\varepsilon}. \nabla a) + f(a \,\mathrm{Tr}\, \boldsymbol{\kappa} + z\Delta a) \Big) a^* = \nabla(z f \nabla a \, a^*) - f z \nabla a. \nabla a^* + z f \,\mathrm{Tr}\, \boldsymbol{\kappa} \, a \, a^*. \tag{66}$$

Combining the derivation of equilibrium equations and the consistency condition for the propagation, we obtain:

Theorem 2. *The solution of the problem of evolution is given by the variational inequality*

$$\frac{\partial \mathcal{F}}{\partial \underline{\mathrm{v}}}.(\underline{\mathrm{v}}^* - \underline{\mathrm{v}}) + \frac{\partial \mathcal{F}}{\partial a}(a^* - a) \geq 0, \tag{67}$$

where

$$\mathcal{F}(\underline{\mathrm{v}}, a) = \int_{\Omega} \frac{1}{2} \boldsymbol{\varepsilon}(\underline{\mathrm{v}}).\mathbb{C}(\mathrm{d}) : \boldsymbol{\varepsilon}(\underline{\mathrm{v}}) \, \mathrm{d}\Omega - \int_{\Omega_c} a \, \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\underline{\mathrm{v}}) \, \mathrm{d}\Omega$$
$$+ \int_{\Omega_c} \frac{1}{2} f(z \|\nabla a\|^2 - \mathrm{Tr}\, \boldsymbol{\kappa} \, a^2) - \frac{1}{2} \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \frac{\partial \boldsymbol{\varepsilon}}{\partial z} a^2 \, \mathrm{d}\Omega. \tag{68}$$

The variations with respect to $\underline{\mathrm{v}}$ implies the derivation of 56 with respect to time

$$\begin{cases} \dot{\boldsymbol{\sigma}} & = \mathbb{C}(0) : \boldsymbol{\varepsilon}(\underline{\mathrm{v}}), \text{ in } \Omega_o, \\ \dot{\boldsymbol{\sigma}} & = \mathbb{C}(\mathrm{d}) : \boldsymbol{\varepsilon}(\underline{\mathrm{v}}) - a \dfrac{\partial \mathbb{C}(\mathrm{d})}{\partial \mathrm{d}} : \boldsymbol{\varepsilon}(\underline{u}) \nabla \mathrm{d}. \underline{\nu}, \text{ in } \Omega_c, \\ \dot{\boldsymbol{\sigma}} & = \mathbb{C}(1) : \boldsymbol{\varepsilon}(\underline{\mathrm{v}}), \text{ in } \Omega_1, \end{cases} \tag{69}$$

and the conservation of the momentum:

$$\mathrm{div}\, \dot{\boldsymbol{\sigma}} = 0, \text{ over } \Omega, \quad \dot{\boldsymbol{\sigma}}.\underline{n} = \underline{\dot{T}}^d, \text{ along } \partial\Omega_T. \tag{70}$$

The variations relatively to $a(s^\alpha)$ gives

$$\int_{\Omega_c} -\frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}\, \delta a + f(z\nabla a.\nabla\delta a - \mathrm{Tr}\,\boldsymbol{\kappa}\; a\; \delta a) - \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \frac{\partial \boldsymbol{\varepsilon}}{\partial z} a\; \delta a\; \mathrm{d}\Omega. \tag{71}$$

By integration by part we obtain

$$\int_{\Omega_c} -\frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}\, \delta a + \nabla(f\; z\; \nabla a\; \delta a) - f\; z\Delta a\; \delta a - \mathrm{Tr}\,\boldsymbol{\kappa}\; a\; \delta a - \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \frac{\partial \boldsymbol{\varepsilon}}{\partial z}\; a\; \delta a\; \mathrm{d}\Omega. \tag{72}$$

And using the boundary condition $\delta a = 0$ along $\partial\Gamma_o^+$, we recover the consistency condition

$$\int_{\Gamma_o}\int_o^{l_c} \Big(\frac{\partial f}{\partial \boldsymbol{\varepsilon}} : D_a\boldsymbol{\varepsilon} + f\,\mathrm{Tr}(\nabla D_a \mathbf{M})\Big)\; j(z)\mathrm{d}z\; \delta a\; \mathrm{d}S. \tag{73}$$

During the phase of initiation of the layer, some additional terms due to $\dot{l}$ are present.

Comparing to the case of sharp interface, the velocity must be more regular, due to the presence of its tangential gradient. The curvature plays a fundamental role on the stability and bifurcation of the solution.

5. Some Examples

5.1. A cylinder and a sphere under radial expansion

To illustrate the preceding results we propose to consider the case of a composite cylinder or a composite sphere, with a kernel of material 1 and a core of material 0 in the case of sharp interface, the radius of the boundary Γ_o is R_o, the external radius is R_e. And we consider radial loading. The displacement solution of the problem of equilibrium is assumed to be radial

$$\underline{u} = u(r)\underline{e}_r, \tag{74}$$

and the strain are

- for the cylinder $n = 2$, $\varepsilon_r = \dfrac{\mathrm{d}u}{\mathrm{d}r}, \varepsilon_\theta = \dfrac{u}{r}, \varepsilon_z = 0,$
- for the sphere $n = 3$, $\varepsilon_r = \dfrac{\mathrm{d}u}{\mathrm{d}r}, \varepsilon_\theta = \varepsilon_\phi = \dfrac{u}{r}.$

In linear elasticity, with Lamés moduli λ, μ the conservation of the momentum, when $\Lambda(r) = \lambda(r) + 2\mu$ with a shear modulus μ uniform, implies the differential equation on $u(r)$:

$$\frac{\mathrm{d}}{\mathrm{d}r}(\Lambda(r)(\frac{\mathrm{d}u}{\mathrm{d}r} + (n-1)\frac{u}{r})) = 0. \tag{75}$$

The displacement solution is then obtained in the composite structure, assuming that the continuity of u at $r = 0$ is satisfied, we have

$$r^{n-1}u(r) = A\int_o^r \frac{r^{n-1}}{\Lambda} dr. \tag{76}$$

The radial displacement for the global response is given by

$$u(R_e) = ER_e, \quad R_e^n E = AI_n(R_e). \tag{77}$$

The total energy is

$$\mathcal{E} = 2^{n-1}\pi \int_o^{R_e} \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, r^{n-1} dr = 2^{n-1}\pi\, \sigma_{rr}(R_e)\, u(R_e) R_e^{n-1}. \tag{78}$$

As the radial stress is

$$\sigma_{rr} = \lambda(\frac{\mathrm{d}u}{\mathrm{d}r} + (n-1)\frac{u}{r} + 2\mu\frac{\mathrm{d}u}{\mathrm{d}r} = A - 2\mu(n-1)\frac{u}{r}, \tag{79}$$

the total strain energy is obtained:

$$\mathcal{E} = 2^{n-1}\pi R_e^{n-1} E^2(\frac{R_e^n}{I_n(R_e)} - 2\mu). \tag{80}$$

5.2. The case of a sharp interface

We consider now a two phase composite, the boundary Γ_o between the sound material and the partially damaged materiel is defined by the surface with radius R_o. When the radius R_o evolves according to the normality rule, potential energy is evolving and the dissipation is determined. For the composite system, the integral I_n depends upon R_o

$$I_n = \int_o^{R_o} \frac{r^{n-1}}{\Lambda_1} dr + \int_{R_o}^{R_e} \frac{r^{n-1}}{\Lambda_o} \mathrm{d}r. \tag{81}$$

For given E, A and I_n satisfies $ER^{n-1} = AI_n$. Combining the derivation of I_n and E relatively to R_o we find the dissipation

$$D_m = -\frac{\partial \mathcal{E}}{\partial R_o} a = 2^{n-1}\pi\, R_o^{n-1}\, \mathcal{G}\, a, \tag{82}$$

where the energy release rate by unit of areas satisfies

$$\mathcal{G} = \mathcal{G}(E, R_o) = \frac{1}{2} A^2 \frac{I_n}{R_o}. \tag{83}$$

During the motion of Γ_o the critical value G_c is conserved $\mathcal{G} = G_c$, then $A = A_c$ with

$$\mathcal{G} = \frac{1}{2} A_c^2 \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} = G_c. \tag{84}$$

5.3. The evolution of the system

Initially the radius is $R_o = R_i$ and then $I_n(R_o) = \dfrac{R_e^n}{n\Lambda_o}(1 + c(R_o)\dfrac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1})$ where c is the volume fraction of material 1 $c(R_o) = (\dfrac{R_o}{R_e})^{n-1}$. For increasing value E of the radial loading, the global response of the composite is decomposed in three state

- State I, $A \leq A_c$ then the answer is purely elastic, $E \leq E_c(c(R_i))$

$$\Sigma = \sigma_{rr}(R_e) = A - 2^{n-1}\mu E, \quad E = A\frac{I_n(R_i)}{R_e^n}, \tag{85}$$

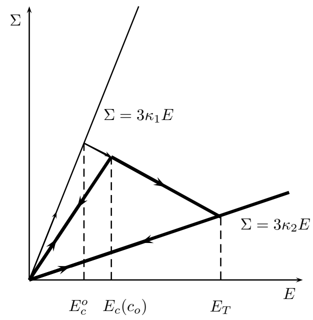

Figure 2. The global response on a composite sphere under radial expansion.

- State II. The interface is moving, now $A = A_c$, the volume fraction evolves with R_o and

$$ER_e^n = A_c I_n(R_o), \quad \Sigma = \Lambda_o A_c - 2^{n-1}\mu E \tag{86}$$

From this state, the relation between E and Σ is linear and Σ is a decreasing function of E.

When R_i tends to 0, the critical value $E_c(c(R_i))$ tends to E_c^o which corresponds to the critical value for the propagation of an infinitesimal defect as proposed in [11].

5.4. The layer model

To describe a continuous transition, we consider the damage law $\mathrm{d}(\phi) = \dfrac{\phi}{l_c}$ and

$$\frac{1}{\Lambda} = \frac{d}{\Lambda_1} + \frac{1-d}{\Lambda_o} \tag{87}$$

The values Λ_o, Λ_1 are those used for the sharp interface. Initially the damaged material appears at $r = 0$, and R_o increases from 0 to l_c. After that the layer i moving keeping its thickness constant.

- State 0: Initially the response of the composite structure is purely elastic. When a critical value is reached, the transition zone is established.
- State I : Initiation of the transition zone. During the initiation of the layer, $\phi = (R_o - r)/l_c, r \leq l_c$, and the integral I_n is given by

$$I_n = \int_o^{R_e} \frac{r^{n-1}}{\Lambda(r)} dr = \frac{R_e^n}{n\Lambda_o} + \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \frac{R_o^{n+1}}{n(n+1)l_c}. \tag{88}$$

During this state, the dissipation satisfies the relation

$$D_m = 2^{n-1}\pi \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \frac{A^2}{l_c} \frac{R_o^n}{n} = 2^{n-1}\pi Y_c \int_o^{R_o} \frac{r^{n-1}}{l_c} dr, \tag{89}$$

and then

$$\frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} A_c^2 = Y_c. \tag{90}$$

As for the case of sharp layer the parameter $A = A_c$ is constant. The value A_c determines the critical strain E_c for the beginning of a kernel of damaged material: $E_c = A_c I_n(0)/R_e^n$.

During the phase of initiation of the layer $A = A_c$, $R_o \leq l_c$ and

$$\Sigma = A_c - 2^{n-1}\mu E, \quad E = A_c I_n(R_o)/R_e^n. \tag{91}$$

After a purely elastic response during the increase of E from 0 to E_c, the global stress decreases linearly with E.

- State II : The finite thickness is reached, $R_o = l_c$. From now, the layer is moving inside the structure keeping its thickness constant. d $= (R_o - r)/l_c, R_o - l_c \leq rR_o$. The integral I_n becomes

$$In(R_o) = \frac{R_e^n}{n\Lambda_o} + \frac{\Lambda_o - \Lambda_1}{\Lambda_o\Lambda_1}\frac{1}{l_c n(n+1)}(R_o^{n+1} - (R_o - l_c)^{n+1}). \tag{92}$$

The dissipation during the propagation gives the constrain

$$D_m = 2^{n-1}\pi\frac{\Lambda_o - \Lambda_1}{\Lambda_o\Lambda_1}\frac{A^2}{nl_c}(R_o^n - (R_o - l_c)^n) = 2^{n-1}\pi Y_c\int_o^{R_o}\frac{r^{n-1}}{l_c}dr. \tag{93}$$

After simplification we recover the value $A = A_c$. Then after identification $Y_c = G_c$, the response with a moving layer cannot be distinct from the sharp interface. Such a result is observed for the uni-axial case, like a bar in tension, for any constitutive law [13]. This is due for the particular choice of the function $\Lambda(d)$.

For the same function d(ϕ) we can choose Λ(d) = d Λ_1 + (1 − d) Λ_o. During the propagation of the layer when thickness is l_c, we have

$$\Lambda(r) = \Lambda_1\frac{R_o - r}{l_c} + (1 - \frac{R_o - r}{l_c})\Lambda_o = \Lambda_M + \Delta\Lambda\frac{r}{l_c}. \tag{94}$$

For this behaviour, we have for the cylinder

$$I_2 = \frac{(R_o - l_c)^2}{2\Lambda_1} + \frac{R_e^2 - R_o^2}{2\Lambda_o} + \frac{l_c^2}{\Delta\Lambda} - \frac{\Lambda_M l_c^2}{(\Delta\Lambda)^2}\log\frac{\Lambda_o}{\Lambda_1}, \tag{95}$$

and the constrain due to the normality law becomes

$$A^2\frac{\Lambda_o - \Lambda_1}{\Lambda_o\Lambda_1}(R_o + \frac{l_c\Lambda_o}{\Delta\Lambda}) = 2Y_c R_o. \tag{96}$$

In this case, the influence of the layer curvature is emphasized. It can be noticed that when l_c/R_o tends to zero, the result for the sharp interface is recovered.

5.5. Example of bifurcation of solution

We consider now a model of blister test. Similar study is made in [18]. The damage parameter d = 0 for the bonded part and d = 1 for the domain of a membrane peeled of a rigid substrate.

The membrane is submitted to an internal pressure p and the deformation of the membrane is described by the vertical displacement u relatively to the substrate as shown in the figure. The domain Ω_1 is now a surface S where the membrane is peeled and the potential energy takes the form

$$\mathcal{E}(u, S, p) = \int_S \frac{1}{2}K\nabla u^2 - pu \, \mathrm{d}S. \tag{97}$$

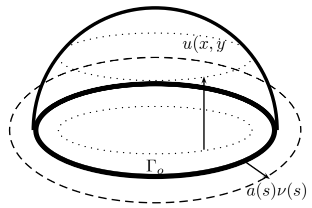

Figure 3. Geometry of a inflated membrane.

On ∂S the membrane is bonded then $u(x,t) = 0,\ x \in \partial S$ Under the internal pressure, the membrane is deformed and the equilibrium state is given by the minimum of the potential energy

$$\frac{\partial \mathcal{E}}{\partial u}.\delta u = \int_S K\nabla u \nabla \delta u - p\delta u \ \mathrm{d}S \ = 0, \tag{98}$$

for all δu defined on S and $\delta u(x) = 0,\ \in \partial S$. We obtain

$$K\Delta u + p = 0, \quad \text{over } S, \quad u = 0, \text{ along } \partial S. \tag{99}$$

For a variation of the loading $\dot{p}$, the boundary can move and the boundary condition $u = 0$ is conserved following the motion of ∂S with normal velocity a. The rate of the displacement for a point on the boundary satisfies the Hadamard relation

$$v + a\nabla u.\underline{n} = 0, \tag{100}$$

and the variation of potential energy satisfies simultaneously

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \int_S K\nabla u.\nabla v - pv - \dot{p}u \ \mathrm{d}S \ + \int_{\partial S} (\frac{1}{2}K\nabla u^2 - pu)\phi \ ds. \tag{101}$$

The problem of evolution of a state of equilibrium is given by

$$K\Delta v + \dot{p} = 0, \text{ over } S, \quad v + a\nabla u.\underline{n} = 0, \text{ along } \partial S. \tag{102}$$

The evolution of $\mathcal{E}$ contains two terms, one due to v and another due to a The variation relatively to a defined the energy release rate

$$-\frac{\partial \mathcal{E}}{\partial S}\delta S \ = \int_{\partial S} \mathcal{G}(s)a(s) \ \mathrm{d}S, \quad \mathcal{G}(s) = \frac{1}{2}K(\nabla u.\underline{n})^2. \tag{103}$$

The propagation law is defined by the normality rule

$$\mathcal{G} \leq G_c, \quad a \geq 0, \quad a(\mathcal{G} - G_c) = 0. \tag{104}$$

The convected derivative of $\mathcal{G}$ is easy to calculate

$$D_a\mathcal{G} = K\nabla u.\underline{n}(\nabla v.\underline{n} + \phi\nabla\nabla u\underline{n}.\underline{n}). \tag{105}$$

As $D_a u = v + a\nabla u.\underline{n} = 0$, then any variation of u satisfies $\delta u + \delta a\nabla u.\underline{n} = 0$.

The functional $\mathcal{F}(v, a)$ is reduced to

$$\mathcal{F} = \int_S \frac{1}{2} K \nabla v . \nabla v - \dot{p} v \, \mathrm{d}S - \int_\Gamma \frac{1}{2} a^2 K \nabla u . \underline{n} \, \nabla \nabla u . \underline{n} . \underline{n} \, ds. \tag{106}$$

For an initial circular geometry S with radius R, the solution of the problem is $u = -p/4K(r^2 - R^2)$ and $\mathcal{G}(R) \leq G_c$ gives a critical pressure

$$pR < p_c R = 2\sigma_o. \tag{107}$$

For p pressure greater than the critical value domain S varies according to the normality rule. The solution satisfies

$$\frac{\partial \mathcal{F}}{\partial v}(v^* - v) + \frac{\partial \mathcal{F}}{\partial \phi}(a^* - a) \geq 0, \quad \forall a^* \geq 0. \tag{108}$$

For given distribution of local speed $a(s)$, the solution of the problem is $v = v(\dot{p}, a)$, we can introduce the functional Q

$$Q(a) = \mathcal{F}(\underline{\mathrm{v}}(\dot{p}, a), a). \tag{109}$$

To determine the set of possible solutions, solution $v(\dot{p}, a)$ is obtained by considering that the propagation speed a is developed according to the expansion $a = \alpha_o + \sum_i \alpha_i \cos(i\theta) + \beta_i \sin(i\theta)$.

- For prescribed pressure p we have

$$v = \frac{pr}{2K}(\alpha_o + \sum_i (\alpha_i \cos(i\theta) + \beta_i \sin(i\theta))(\frac{r}{R})^i), \tag{110}$$

  the reduced functional

$$Q(a) = 2\pi G_c(-2\alpha_o^2 + \sum_i (i-1)(\alpha_i^2 + \beta_i^2)), \tag{111}$$

  is not positive definite, then the circular geometry is unstable.
- But when we control the volume the pressure varies according to the motion of Γ_o:

$$v = \frac{pr}{2K}(\alpha_o + \sum_i (\alpha_i \cos(i\theta) + \beta_i \sin(i\theta))(\frac{r}{R})^i) + \delta p \frac{R^2 - r^2}{4K}, \quad \int_S v \mathrm{d}s = 0. \tag{112}$$

  The variations of pressure is determined $\delta p = -\frac{4p}{R}\alpha_o$ and the functional is now

$$Q(a) = 2\pi G_c(6\alpha_o^2 + \sum_i (i-1)(\alpha_i^2 + \beta_i^2)). \tag{113}$$

  The positivity of Q for $a \geq 0$ is obtained, solutions $a = \frac{\dot{R}}{R} + \alpha_1 \cos\theta + \beta_1 \sin\theta \geq 0$ are possible for sufficiently small α_1, β_1, the stability of the circular geometry is ensured. The functional Q is not strictly positive for all a and possible bifurcation, loss of circular geometry is obtained.

The discussion of delamination of laminates can be studied in this framework as proposed in [19, 20, 21]. The model of description of the laminate structure has a strong influence on the condition of stability and bifurcation of solution [22, 23].

6. A Local Model of Rupture

Neuber [24] has investigated the mechanism of stress-concentration near a notch and crack propagation. A solution is obtained for any non-linear stress-strain laws monotonically increasing and all loading intensities. The boundary of the notch consists of two parallel straight lines and a cycloid along which the amount shear is uniform. Bui and Ehrlacher [6] solved the same problem for a linear elastic material; the thickness of the damaged zone is determined as a function of the stress intensity factor of the equivalent crack. Similar solution are obtained also in elastoplasticity by Bui [5] and for crack in [25]. More recently, the results have been generalized to different families of non-linear elastic brittle material [26] using the hodograph transformation proposed by [27, 28]. To determine the mechanical quantities over the domain Ω, the hodograph transformation is used [27, 29]. Another technique is also available [24].

In the transformation of hodograph, the components of the gradient ∇w become the new independent variables.

$$(x_1, x_2) \to (\xi_1, \xi_2), \quad \xi_\alpha = w_{,\alpha}(x_1, x_2). \tag{114}$$

The displacement appears as a potential ($dw = \xi_\alpha dx_\alpha$). The mapping is invertible provided the Jacobian $H = w_{,11}w_{,22} - (w_{,12})^2$ does not vanish. Denote by U the Legendre transformation of w with respect to ξ_α

$$U(\xi_1, \xi_2) = x_\alpha w_{,\alpha}(x_1, x_2) - w(x_1, x_2). \tag{115}$$

By differentiating U with respect to ξ, the conjugate equations are obtained

$$x_\alpha = \frac{\partial U}{\partial \xi_\alpha}, \quad w = \xi.\nabla U - U. \tag{116}$$

In the hodograph plane, the polar coordinates are used ($\xi_1 = R\cos\Theta, \xi_2 = R\sin\Theta$) then this inverse of the mapping is given by:

$$x_1 = \cos\Theta\frac{\partial U}{\partial R} - \frac{\sin\Theta}{R}\frac{\partial U}{\partial \Theta}, \quad x_2 = \sin\Theta\frac{\partial U}{\partial R} + \frac{\cos\Theta}{R}\frac{\partial U}{\partial \Theta}. \tag{117}$$

The stress field satisfies:

$$\tau_1 = \boldsymbol{\sigma}_{13} = \mu(R)R\cos\Theta, \quad \tau_2 = \boldsymbol{\sigma}_{23} = \mu(R)R\sin\Theta. \tag{118}$$

The equation of motion is rewritten in the hodograph plane (R, Θ) using the mapping 117. The notation $()'$ indicates a derivation with respect to R. The differential equation obtained in statics takes the expression

$$\frac{\partial}{\partial R}\Big(\mu(R)R\frac{\partial U}{\partial R}\Big) + \frac{(\mu(R)R)'}{R}\frac{\partial^2 U}{\partial \Theta^2} = 0. \tag{119}$$

This equation is homogeneous of degree one in U. Using 115 the displacement w is

$$w = R^2\frac{\partial}{\partial R}(\frac{U}{R}). \tag{120}$$

For the considered example, the geometry of the crack is the straight line $-\infty \le x_1 \le 0$. The direction of the line is $\underline{e}_1$. The applied loading is that of a crack obtained in classical linearised elasticity or

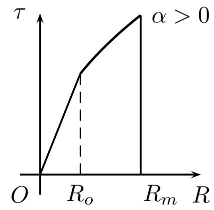

Figure 4. The non linear elastic brittle material.

embedded in neo-Hookean material. For an inner point of view, the local behaviour is non-linear and the amount of shear is limited by a critical value R_m, as described by the constitutive law:

$$\begin{cases} R \leq R_o, & \tau = \mu_o R, \\ R_o \leq R \leq R_m, & \tau = \mu_o R_o (\dfrac{R}{R_o})^\alpha = \hat{\mu} R^\alpha, \\ R_m \leq R, & \tau = 0. \end{cases} \tag{121}$$

Then for $R \leq R_o$ the differential equation is elliptic, for $R_m \geq R \geq R_o$ ellipticity is ensured if $\alpha \geq 0$ otherwise the differential equation becomes hyperbolic. We only consider here the ellipticity case.

For the considered constitutive law, solution in the hodograph plane is build by the combination of solution for the linear part of the constitutive law and solution for the non linear part with the help of peculiar solutions. The potential U is searched as [26, 30]:

$$\begin{cases} 0 \leq R \leq R_o, \quad U = U_o(R, \Theta), \\ U_o = A_o R \cos\Theta \displaystyle\int_R^{R_o} \frac{\mathrm{d}t}{\mu_o t^3} - B_o R(\log R \cos\Theta - \Theta \sin\Theta) + C_o R \cos\Theta, \\ R_o \leq R \leq R_m, \quad U_1 = \hat{U}(R, \Theta), \\ U_1 = A_1 R \cos\Theta \displaystyle\int_R^{R_o} \frac{\mathrm{d}t}{\hat{\mu}_o t^{\alpha+2}} - B_1 R(\frac{2\alpha}{\alpha+1} \log R \cos\Theta - \Theta \sin\Theta) + C_1 R \cos\Theta. \end{cases} \tag{122}$$

The constants $(A_o, A_1, B_o, B_1, C_o, C_1)$ are determined by:

- (a) continuity conditions of the potential in $R = R_o$,
- (b) continuity of the displacement w everywhere,
- (c) asymptotic behaviour when $R \to 0$,
- (d) traction free- boundary conditions along the boundary of the damaged zone.

6.1. Determination of the constants

- (a) : Continuity of the potential implies that $B_o = B_1 = B$, and the relation

$$B \frac{2\alpha}{\alpha+1} R_o \log R_o + C_1 R_o = B R_o \log R_o + C_o R_o. \tag{123}$$

Then the displacement is given by

$$\begin{cases} 0 \le R \le R_o, & w = w_o(R,\Theta) = -(\frac{A_o}{\mu_o R} + BR)\cos\Theta, \\ R_o \le R \le R_m, & w = \hat{w}(R,\Theta) = -(\frac{A_1}{\hat{\mu}_o R^\alpha} + \frac{2B\alpha}{\alpha+1}R)\cos\Theta. \end{cases} \tag{124}$$

- (b) Displacement is continuous en $R = R_o$

$$A_1 + \mu_o R_o^2 B\frac{\alpha-1}{\alpha+1} = A_o. \tag{125}$$

- (c) Displacement satisfies the matching condition at ∞, so we obtain

$$A_o = \frac{K^2}{\pi\mu_o}. \tag{126}$$

- (d) The boundary along the damaged zone is traction free.

To simplify the expression, we adopt an adimensional formulation. Let us define new parameters (a, a_1, b)

$$\frac{A_o}{\mu_o R_o^2} = \frac{K^2}{\pi\tau_o^2} = a, \quad \frac{A_1}{\mu_o R_o^2} = aa_1, \quad B = ba. \tag{127}$$

We adopt $C_o = 2 - B + B\log(R_o)$ in order to determine the position of the quasi-crack along $x_2 = 0$, and we consider the notation $X_i = \frac{x_i}{a}, \rho = \frac{R}{R_o}$. Then $\rho_o = 1, \rho_m = \frac{R_m}{R_o}$.

The condition (126) is then rewritten as

$$a_1 + b\frac{\alpha-1}{\alpha+1} = 1. \tag{128}$$

With these notations, the traction-free boundary condition (d) is now explained.

For that, the image of the hodograph plane is determined.

- $0 \le \rho \le 1$,

$$\begin{aligned} X_1 &= -\frac{1}{2} - \frac{1}{2\rho^2}\cos(2\Theta) - b\log\rho, \\ X_2 &= -\frac{1}{2\rho^2}\sin(2\Theta) + b(\Theta - \frac{\pi}{2}), \\ W &= -(\frac{1}{\rho} + b\rho)\cos\Theta. \end{aligned}$$

- $1 \le \rho \le \rho_m$,

$$\begin{aligned} X_1 &= -(\frac{a_1}{\rho^{\alpha+1}} + b\frac{\alpha-1}{\alpha+1})\frac{\cos 2\Theta}{2} - (\frac{a_1}{\rho^{\alpha+1}} + b)\frac{\alpha-1}{2(\alpha+1)} - \frac{2b\alpha}{\alpha+1}\log\rho - \frac{a_1}{\alpha+1}, \\ X_2 &= -(\frac{a_1}{\rho^{\alpha+1}} + b\frac{\alpha-1}{\alpha+1})\frac{\sin 2\Theta}{2} + b(\Theta - \frac{\pi}{2}), \\ W &= -(\frac{a_1}{\rho^\alpha} + \frac{2\alpha}{\alpha+1}b\rho)\cos\Theta. \end{aligned}$$

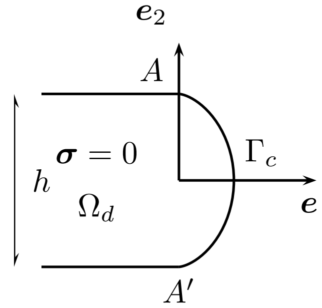

Figure 5. The geometry of a quasi-crack.

We have moved the frame along $\underline{e}_2$ with a shift of $-b\pi/2$, to emphasize the symmetry of the geometry with respect to $\underline{e}_1$. For $\rho = \rho_a$, the curve $X_1(\rho_a, \Theta), X_2(\rho_a, \Theta)$ is a cycloid. With this geometry, the traction- free boundary condition is given by the equation

$$\tau_1 dX_2 - \tau_2 dX_1 = 0. \tag{129}$$

The boundary is decomposed into three parts: the two horizontal lines $\Theta = \pi/2 \pm \pi/2$ and the cycloid where $\rho = \rho_m$. The traction-free boundary condition is satisfied for $\Theta = \pi/2 \pm \pi/2$ for all ρ, and implies a relation along the cycloid where $\rho = \rho_m$:

$$(\frac{2b}{\alpha + 1} - \frac{a_1}{\rho_m^{\alpha+1}}) \cos\Theta = 0. \tag{130}$$

This relation can be combined with (126), then

$$1 = \frac{2b}{\alpha + 1}(\rho_m^{\alpha+1} + \frac{\alpha - 1}{2}) \tag{131}$$

It can be noticed that the thickness H of the damaged zone is $H = ba\pi$.

For $\rho_m \to \infty$, the quantity b vanishes, the quasi-crack tends to the classical crack, the thickness H vanishes.

6.2. Connection with release rate of energy

The strain energy for the critical shear R_m is given by

$$\mathcal{W}(R_m) = \int_o^{R_m} \tau(t)\, \mathrm{d}t = \frac{1}{\mu_o}\frac{\tau_o^2}{\alpha + 1}(\rho_m^{\alpha+1} + \frac{\alpha - 1}{2}). \tag{132}$$

As a is a function of the equivalent loading K the relation 131 determines the thickness of the damaged zone

$$\frac{1}{2}a\pi = \frac{h}{\alpha + 1}(\rho_m^{\alpha+1} + \frac{\alpha - 1}{2}) = \frac{1}{2}\frac{K^2}{\tau_o^2}. \tag{133}$$

The flux of the energy along the cycloid is given by

$$D_d = h\mathcal{W}(R_m) = \pi a \frac{\tau_o^2}{2\mu_o} = \frac{K^2}{2\mu_o}. \tag{134}$$

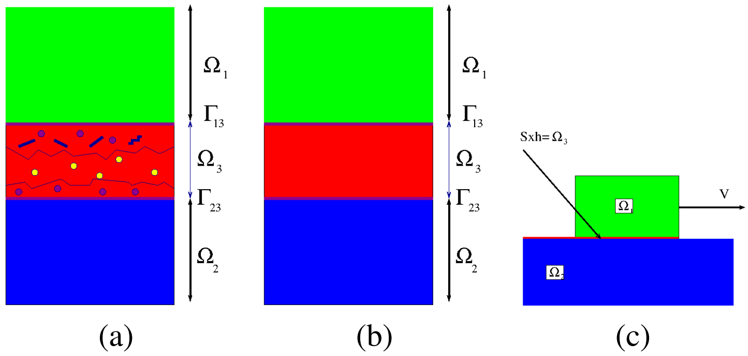

Figure 6. A global approach of wear contact : (a) micromecanisms, (b) homogeneisation of Ω_3, (c) Interface layer behaviour by unit area of contact.

This flux is exactly the energy release rate for the equivalent crack given by the condition of loading at infinity

$$\mathcal{G} = J = h\,\mathcal{W}(R_m). \tag{135}$$

For different constitutive relations, a numerical scheme is proposed in [31] to solve the problem of propagation of a quasi-crack for some specific classes of non linear elastic brittle materials.

7. Modelling of Contact Wear

The system consist of two sliding and contacting bodies Ω_1 and Ω_2 separated by a contact interface Ω_3. The mechanical behaviour of the two bodies are known. The behaviour of the interface Ω_3 is more complex. Particles are detached from sound solids when some local criteria are satisfied at the boundary. Wear leads to geometrical changes and modification of contact conditions. Wear debris induce a specific layer. The interface is a complex medium made of particles; eventually of a lubricant fluid and of damaged zones inside the two bodies near the surface of contact. The interface here is considered at a macroscopic level as an homogeneous body obtained by an averaging process through the thickness $H = 2h$ of Ω_3. This thickness is so small compared to the size of the contact zone and of the tribological system that the condition of such homogeneity can be acceptable.

During wear process, the boundaries $\Gamma_1 = \partial\Omega_1 \cap \partial\Omega_3$ and $\Gamma_2 = \partial\Omega_2 \cap \partial\Omega_3$ evolve. During this process a flux of matter is lost from the materials and the framework developed previously can be applied. Along the two interfaces Γ_i energy is dissipated due to the motion of the boundaries

$$D_i = \int_{\Gamma_i} a_i[\psi]_\Gamma - \underline{n}_i.\boldsymbol{\sigma}.[\underline{\mathrm{v}}]_\Gamma \,\mathrm{d}S = \int_{\Gamma_i} \mathcal{G}_i\, a_i \mathrm{d}s, \tag{136}$$

and inside the volume of the layer Ω_3 a volume dissipation d_m due to damage, plasticity or viscosity, is also present. By integration over the thickness of the layer Ω_3, dissipation by unit of contact area is defined as

$$D_m = \sum_i \mathcal{G}_i j_i a_i + \int_H d_m(z)\, j(z)\mathrm{d}z. \tag{137}$$

A point of the layer Ω_3 is defined relatively to middle surface $S(X,t) = 0$ with normal $\underline{n}$. Then a point of the layer has coordinates (X, z) such that $x = X + z\underline{n}(X)$. For $x_1 \in \Gamma_1, z_1 = h$ and for $x_2 \in \Gamma_2, z_2 = -h$. The unit area of surface at distance z from $S(X,t) = 0$ are $\mathrm{d}S(z) = j(z)\,\mathrm{d}S$ and $j_1 = j(h), j_2 = j(-h)$.

The two first terms of the dissipation are associated to the lost of material. The last term can be interpreted as local friction [8]

$$\text{friction } D_3 = \int_H d_m\, dz = \int_H (\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi}_3)\, dz. \tag{138}$$

The main difficulty of the model is to choose the behaviour of the interface which takes into account of wear debris.

7.1. Interface study

All mechanical quantities are defined with respect to the middle surface : $S(X,t) = 0$ with normal $\underline{n}$. And Γ_1, Γ_2 are defined by

$$x_1 = X + h(X,t)\underline{\nu}, \quad x_2 = X - h(X,t)\underline{\nu}. \tag{139}$$

We denote a the velocity of S.

$$D_a(x_1) = a_1\underline{\nu}_1, \quad D_a(x_2) = a_2\underline{\nu}_2. \tag{140}$$

The displacement along the boundary Γ_i is continuous

$$[\underline{u}(X \pm h(X,t)\underline{n}, t)]_{\Gamma_i} = 0, \tag{141}$$

and also the stress vector $[\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_3].\underline{\nu} = 0$.

The internal state of the layer is defined by the total strain energy by unit of contact area

$$\psi_S(\underline{u}_1, \underline{u}_2, \alpha) = \frac{1}{\rho_S}\int_{-h}^{h} \rho(x,z)\psi_3(\boldsymbol{\varepsilon}(x + z\underline{\nu}), \alpha)\, j(x,z) dz, \tag{142}$$

the surface density of mass

$$\rho_S = \int_{-h}^{h} \rho(x,z)\; j(x,z)\mathrm{d}z \tag{143}$$

the total potential of dissipation

$$D_S(\underline{\mathrm{v}}_1, \underline{\mathrm{v}}_2, \dot{\alpha}) = \frac{1}{\rho_S}\int_{-h}^{h} \rho(x,z)\; d_m\Big(\dot{\boldsymbol{\varepsilon}}(x + z\underline{\nu}), \dot{\alpha}\Big)\; j(x,z)\mathrm{d}z \tag{144}$$

To build model of interface, the displacement inside the layer is decomposed as an expansion relatively to z.

$$\underline{u} = \underline{u}_o + z\underline{u}^1 + z^2\underline{u}^2 + ... \tag{145}$$

Up to order 1 we have

$$\underline{u}_1 = \underline{u}(X - h\underline{\nu}), \quad \underline{u}_2 = \underline{u}(X + h\underline{\nu}) \tag{146}$$

and

$$\underline{u}_1 - \underline{u}_2 = \nabla\underline{u}(X,0).\underline{\nu}h = \underline{u}^1 h, \quad \underline{u}_o = \frac{1}{2}(\underline{u}_1 + \underline{u}_2) \tag{147}$$

It can be shown that the equilibrium along the interface satisfies

$$\boldsymbol{\sigma}.\underline{n}_i = \rho_S\Big(\frac{\partial\psi_S}{\partial\underline{u}_i} + \frac{\partial D_S}{\partial\underline{\mathrm{v}}_i}\Big). \tag{148}$$

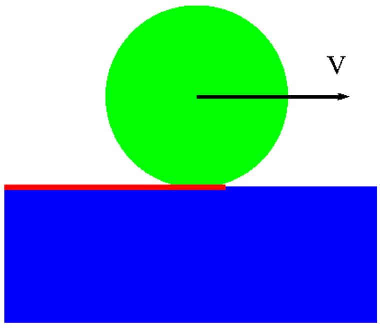

Figure 7. A rigid punch on an half-elastic plane.

7.2. Sliding contact in steady relative motion

The rigid punch has a vertical displacement $u_y^p = \delta + x^2/2R$ and we assume that wear occurs only in the half space. Ahead the punch there is no debris, then the volume fraction of debris $f = 0$. The thickness of the interface is H_o, which corresponds to the sum of roughness of the solids and contains the thickness of the incompressible fluid. Due to wear, the thickness evolves. The mass conservation and the fluid incompressibility give the relations between the wear rate $\phi(x)$, the fraction of debris $f(x)$ and the thickness of the thin layer $X(x)$ [32]. All the equations of conservation are written in the moving frame with the punch at the velocity $V\underline{e}_x$.

The free energy of the mixture is given by

$$\psi_s(w, f) = k(f)\frac{1}{2}(w_n)^2 + k_t(f)(w_t - \alpha_t)^2. \tag{149}$$

A potential of dissipation is given to determine the irreversible contribution, essentially due to viscosity

$$d(\dot{w}, \dot{\alpha}_t) = \frac{1}{2}\eta_n(f)\dot{w}_n^2 + \frac{1}{2}\eta_t(f)\dot{w}_t^2 + \frac{1}{2}\eta_a(f)\dot{\alpha}_t^2. \tag{150}$$

Then the local state equations are

$$\underline{n}.\boldsymbol{\sigma} = k(f)w_n\underline{n} + k_t(f)(w_t - \alpha_t)\underline{\tau} + \eta_n\dot{w}_n\underline{n} + \eta_t\dot{w}_t\underline{\tau}, \quad A = k_t(f)(w_t - \alpha_t) = \eta_a(f)\dot{\alpha}_t. \tag{151}$$

where$A$ is the driving force associated to viscosity. This constitutive law generalizes the law use in ([32]) in which $\eta_a = k_t = 0$. For ($\eta_a = k_t = 0$) we have an interface behaviour given by

$$\sigma_{yy} = k(f)w_n, \quad \sigma_{xy} = \eta_t(f)\dot{w}_t, \tag{152}$$

$k(f)$ and $\eta(f)$ are chosen from typical homogenized value of the phases.

$$\frac{1}{k(f)} = \frac{f}{K_s} + \frac{(1-f)}{K_f}, \quad \eta_t = \eta_o(1 + 2.5f), \tag{153}$$

that the homogenized Reuss's model for the stiffness and the Einstein's law for the viscosity.

Introducing these equations in the equilibrium equation determines for a given profile $\phi(x)$ the answer of the elastic half space. The wear rate ϕ must satisfy a complementary law as proposed before. For the sake of simplicity we take

$$\phi = k_p\sigma_{yy}. \tag{154}$$

The half plane has an elastic linear behaviour. In plane strain, the displacement (u_x, u_y) of the interface Γ_2 is given by solving the Galin's equations:

$$co_1 u_{x,x}(x) = co_2 \sigma_{yy} + Vp \frac{1}{\pi} \int_{-a}^{a} \frac{\sigma_{xy}(s)}{s-x} \mathrm{d}s, \tag{155}$$

$$co_1 u_{y,x}(x) = -co_2 \sigma_{xy} + Vp \frac{1}{\pi} \int_{-a}^{a} \frac{\sigma_{yy}(s)}{s-x} \mathrm{d}s, \tag{156}$$

where σ_{yy}, σ_{xy} are the component of the stress vector applied on the half plane, and co_i are coefficients given in terms of Young's modulus E and Poisson's ratio ν. Vp is principal value of integral in sense of Cauchy.

$$co_1 = \frac{E}{2(1-\nu^2)}, \quad co_2 = \frac{1-2\nu}{2(1-\nu)} \tag{157}$$

The solution is obtained analytically by an asymptotic expansion in series of the volume fraction f of particles.

- At zero order, the Hertz's solution is recovered.
- At first order, a dependence with f is obtained. Wear occurs, and the profile of the pressure $\sigma_{yy}(x)$ evolves. The presence of viscous fluid induces a displacement of the maximum of pressure like under the dry contact with friction ([33]).

This analytical solution is studied in paper [32].

8. Conclusions

In this article, we propose a framework using the motion of interface or surface to describe particular irreversible processes : damage, fracture, wear, delamination.

The main idea is the change in the strain energy when the degradation process occurs.

The model is based on the definition of the potential energy and to a complementary law using a normality rule based on the driving force associated to the motion of the interface. The evolution of the system is given by the resolution of a variational inequality. This formulation of the rate boundary value problem leads to discussion of stability of the equilibrium configuration and to possible bifurcation from this state.

Some applications of the concept and examples have illustrated the potentiality of the approach.

Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this manuscript.

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