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Research article

Commodity-linked bonds as an innovative financing instrument for African countries to build back better

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Supplementary

Appendix

The Dynkin Operator

Let $t = t_0 + \Delta t$ and assume that the third partial derivative of $J(.)$ is bounded. By applying Taylor's series theorem, the mean value theorem for integrals, and taking the limits as equation (60) becomes:

$$J(w(t_0), P, r, t_0) \equiv \max_{m,\,\omega_1}\Big[\int_{t_0}^{t_1} U(m(t), t)]dt + E(J(w(t_0), P, r, t_0))$$

$$+J_t dt + J_w E(dW) + J_p E(dP) + J_r E(dr) + J_{wp} E(dWdP) + J_{wr} E(dWdr) \quad \text{(A1)}$$

$$+J_{rp} E(drdP) + 0.5 J_{pp} E(dP)^2 + 0.5 J_{rr} E(dr)^2\Big]$$

However, the net foreign debt constraint (Equation (56)) and Equations (1) and (2) give:

$$E(dW) = [\omega_1 W(\alpha_q - \alpha_h + c/Q - c^B/H) + m - Px + W(\alpha_h + c^B/H)]dt \quad \text{(A2)}$$

$$E(dW)^2 = [\omega_1^2 W^2 (\sigma_q - \psi_r)^2 + 2\omega_1 W^2 \psi_r (\sigma_q - \psi_r) + W^2 \psi_r^2$$

$$+2\rho_{pr} W^2 (\psi_p (\omega_1 - \omega_1^2)(\sigma_q - \psi_r) + (1 - \omega_1)\psi_p \psi_r) + (1 - \omega_1)^2 W^2 \psi_p^2] dt \quad \text{(A3)}$$

$$E(P) = \alpha_p P dt \quad \text{(A4)}$$

$$E(dP)^2 = \sigma_p^2 P^2 dt \quad \text{(A5)}$$

$$E(dr) = \kappa(\theta - r) dt \quad \text{(A6)}$$

$$E(dr)^2 = \sigma_r^2 dt \quad \text{(A7)}$$

$$E(dWdP) = [WP\rho_{pr}\sigma_p (\omega_1 (\sigma_q - \psi_r) + \psi_r) + (1 - \omega_1) WP \sigma_p \psi_p] dt \quad \text{(A8)}$$

$$E(dWdr) = [\sigma_r (\omega_1 W (\sigma_q - \psi_r) + W\psi_r) + (1 - \omega_1) W \sigma_r \rho_{pr} \psi_p] dt \quad \text{(A9)}$$

$$E(dPdr) = \rho_{pr} \sigma_p \sigma_r P dt \quad \text{(A10)}$$

Substituting Equations (A2) to (A10) into Equation (A1), and noting that $E(J(w(t_0), P, r, t)) \equiv J(w(t_0), P, r, t)$, the continuous-time version of the Bellman-Dreyfus fundamental optimality equation is obtained, which is of the form:

$$0 \equiv \max_{m,\,\omega_1} [U(m(t), t) + J_t + J_w \omega_1 W (\alpha_q - \alpha_h + c/Q - c^B/H) + m - Px$$

$$+W(\alpha_h + c^B/H)) + J_p (\alpha_p P) + J_r \kappa (\theta - r))$$

$$+J_{wp} \, (WP\rho_{pr}\sigma_p (\omega_1 (\sigma_q - \psi_r) + \psi_r) + (1 - \omega_1) WP \sigma_p \psi_p)$$

$$+0.5 J_{ww} (\omega_1^2 W^2 (\sigma_q - \psi_r)^2 + 2\omega_1 W^2 \psi_r (\sigma_q - \psi_r) + W^2 \psi_r^2$$

$$+2\rho_{pr} W^2 (\psi_p (\omega_1 - \omega_1^2)(\sigma_q - \psi_r) + (1 - \omega_1)\psi_p \psi_r) + (1 - \omega_1)^2 W^2 \psi_p^2)$$

$$+0.5 \sigma_p^2 P^2 J_{pp} + 0.5 \sigma_r^2 J_{rr}] \quad \text{(A11)}$$

In compact form, Equation (A11) can be expressed as:

$$\Phi(m, D, B; W, P, r, t) = U(m(t), t) + L(J),$$

where L is the Dynkin operator over the variables W, P, and r. This operator is defined as:

$$L(J) = J_t + J_w \omega_1 W(\alpha_q - \alpha_h + c/Q - c^B/H) + m - Px$$
$$+W(\alpha_h + c^B/H)) + J_p(\alpha_p P) + J_r \kappa(\theta - r)) + J_{wp}\ (WP\rho_{pr}\sigma_p(\omega_1(\sigma_q - \psi_r) + \psi_r)$$
$$+ (1-\omega_1)WP\sigma_p\psi_p) + 0.5J_{ww}(\omega_1^2 W^2(\sigma_q - \psi_r)^2 + 2\omega_1 W^2\psi_r(\sigma_q - \psi_r) + W^2\psi_r^2$$
$$+2\rho_{pr}W^2(\psi_p(\omega_1 - \omega_1^2)(\sigma_q - \psi_r) + (1-\omega_1)\psi_p\psi_r) + (1-\omega_1)^2 W^2\psi_p^2) + 0.5\sigma_p^2 P^2 J_{pp} +$$
$$0.5\sigma_r^2 J_{rr} \quad \text{(A12)}$$