Metascience in Aerospace

Metascience in Aerospace, 1(4): 371–378.
DOI: 10.3934/mina.2024017
Received date: 22 October 2024
Revised date: 02 December 2024
Accepted date: 10 December 2024
Published date: 20 December 2024

https://www.aimspress.com/journal/mina

Theory article

Cumulative STF coefficients evaluation and validation

Pasynok Sergey*

Federal State Unitary Enterprise "National Research Institute of Physical Technical and Radio Technical Measurements", VNIIFTRI, Mendeleevo, Solnechnogorsk city, Moscow region, 141570, Russia

* **Correspondence:** E-mail: pasynok@vniiftri.ru; Tel: +7-495-660-5725.

Supplementary

Proof of theorem about cumulative STF coefficients

Following [4], take into account the famous formula for STF part of tensor estimation:

$$\tilde{F}_{<N>} = \sum_{l=0}^{[n/2]} a(n,l)\delta_{(j_1 j_2}..\delta_{j_{2l-1}j_{2l}}\tilde{F}_{j_{2l+1},\dots,j_N)k_1k_1\dots k_Lk_L} = \sum_{l=0}^{[n/2]} a(n,l)\delta_{(2L}\tilde{F}_{N-2L)K_lK_l},$$

were $\tilde{F}_{<N>}$ – STF part of symmetric tensor $\tilde{F}_N$, form $\tilde{F}_{<M>}n_M$ and select the first member of it:

$$\tilde{F}_{<M>}n_M = \sum_{l=0}^{[m/2]} a(m,l)\tilde{F}_{M-2l\;k_1k_1\dots k_lk_l}n_{M-2L} = \tilde{F}_M n_M + \sum_{l=1}^{[m/2]} a(m,l)\tilde{F}_{M-2l\;k_1k_1\dots k_lk_l}n_{M-2L}.$$

From this, one can obtain:

$$\tilde{F}_M n_M = \tilde{F}_{<M>}n_M - \sum_{l=1}^{[m/2]} a(m,l)\tilde{F}_{M-2l\;k_1k_1\dots k_lk_l}n_{M-2L} =$$

$$= \tilde{F}_{<M>}n_M - a(m,1)\tilde{F}_{M-2kk}n_{M-2} - \sum_{l=2}^{[m/2]} a(m,l)\tilde{F}_{M-2l\;k_1k_1\dots k_lk_l}n_{M-2L},$$

and after changing indices of summation:

$$\tilde{F}_M n_M = \tilde{F}_{<M>}n_M - a(n,1)\tilde{F}_{M-2kk}n_{M-2} - \sum_{\substack{l'=0\\ l'+2=l}}^{[(m-4)/2]} a(m,l'+2)\tilde{F}_{M-2l'-4\;k_1k_1\dots k_{l'+2}k_{l'+2}}n_{M-2L'-4}.$$

The first member on the right includes STF tensor already. The other members include symmetric tensors of rank m-2 and lower. Now, one can extract the STF part from tensor $\tilde{F}_{M-2kk}$ exactly as it was done for $\tilde{F}_M$, and converting it with n_{M-2}:

$$\tilde{F}_{M-2kk}n_{M-2} = \tilde{F}_{<M-2>kk}n_{M-2} - \sum_{l=1}^{[(m-2)/2]} a(m-2,l)\tilde{F}_{M-2-2l\ kkk_1k_1\ldots k_lk_l}n_{M-2-2L} =$$
$$= \tilde{F}_{<M-2>kk}n_{M-2} - \sum_{\substack{l'=0\\ l'+1=l}}^{[(m-4)/2]} a(m-2,l'+1)\tilde{F}_{M-4-2L'\ k_1k_1\ldots k_{l'+2}k_{l'+2}}n_{M-4-2L}.$$

Next, introduce notation:

$$\tilde{a}(m,0)=1,\ \ \tilde{a}(m,1)=-a(m,1)$$
$$a(m,l,1)=a(m,1)a(m-2,l+1)-a(m,l+2).$$

Uniting the two previous fomulas, introducing similar ones, and taking into account notation, one can obtain:

$$\tilde{F}_M n_M = \tilde{a}(m,0)\tilde{F}_{<M>}n_M + \tilde{a}(m,1)\tilde{F}_{<M-2>kk}n_{M-2} + \left\{\sum_{l=0}^{[(m-4)/2]} a(m,l,1)\tilde{F}_{M-4-2l\ k_1k_1\ldots k_{l+2}k_{l+2}}n_{M-4-2L}\right\}.$$

Now, the coefficients of the first two members are STF tensors and others coefficients are the symmetric tensors of the rank m-4 and lower. Next, introduce notation:

$$\tilde{a}(m,2)=a(m,0,1)=a(m,1)a(m-2,1)-a(m,2),$$
$$a(m,l,2)=a(m,l+1,1)-a(m-4,l+1)a(m,0,1)=$$
$$=a(m,1)a(m-2,l+2)-a(m,l+3)-a(m-4,l+1)\tilde{a}(m,2).$$

Repeating the same operation with expression in curly braces, one can obtain:

$$\tilde{F}_M n_M = \tilde{a}(m,0)\tilde{F}_{<M>}n_M + \tilde{a}(m,1)\tilde{F}_{<M-2>kk}n_{M-2} + \tilde{a}(m,2)\tilde{F}_{<M-4>kk}n_{M-4} +$$
$$+ \sum_{l=0}^{[(m-6)/2]} a(m,l,2)\tilde{F}_{M-6-2lk_1k_1\ldots k_{l+3}k_{l+3}}n_{M-6-2L}.$$

Now, the coefficients of the first three members are STF tensors and other tensor coefficients are the symmetric tensors of the rank m-6 and lower.

Applying this operation again and again, one will obtain sum members with STF tensors and scalar (for even m) or vector (for odd m). As result:

$$\tilde{F}_M n_M = \sum_{l=0}^{\left[\frac{m}{2}\right]} \tilde{a}(m,l)\hat{F}_{i_1\ldots i_{m-2l}}n_{i_1\ldots i_{m-2l}}\ \ ,\ \ \hat{F}_{i_1\ldots i_{m-2l}} = \tilde{F}_{<i_1\ldots i_{m-2l}>k_1k_1\ldots k_lk_l} \tag{S.1}$$

where $\tilde{a}(m,l)$ are as cumulative STF coefficients and are equal:

$$\tilde{a}(m,0)=1,\ \ \tilde{a}(m,1)=-a(m,1), \tag{S.2}$$
$$\tilde{a}(m,2)=a(m,0,1)=a(m,1)a(m-2,1)-a(m,2),$$

$$\tilde{a}(m,l) = a(m,0,l-1),$$

$$a(m,l,1) = a(m,1)a(m-2,l+1) - a(m,l+2),$$

$$a(m,l,k) = a(m,l+1,k-1) - a(m-2k,l+1)a(m,0,k-1), \quad k=2,\ldots,l.$$

However, if one attempts to direct use formulas (S.2) for $l > 2$ using $a(m,l,k) = a(m,l+1,k-1) - a(m-2k,l+1)a(m,0,k-1)$ and calculating $a(m,l,1)$ after that, then some coefficients not will be determined at the moment of evaluation. To overcome this problem it is neccessory to start from formulae for $a(m,l',1)$ with l' which corresponds the given l. If one takes into account that l decreases on 1 when k increases on 1:

$$a(m,l'-1,2) = a(m,l',1) - a(m-4,l')a(m,0,1),$$

$$\ldots$$

$$a(m,l'-i,i+1) = a(m,l'-i+1,i) - a(m-2(i+1),l'-i+1)a(m,0,i),$$

$$\tilde{a}(m,l) = a(m,0,l-1),$$

then he obtains that for given l he has to take $l'=l-2$, that is, one has to start from coefficient $a(m,l-2,1)$ evaluation. So, formulas (4)–(6) are proved. After that, inserting (S.1) in sum (1) and introducing similar ones yields:

$$\sum_{m=0}^{N} \tilde{F}_M n_M = \sum_{m=0}^{N} \sum_{l=0}^{\left[\frac{m}{2}\right]} \tilde{a}(m,l) \tilde{F}_{<i_1 \ldots i_{m-2l}>k_1k_1 \ldots k_lk_l} n_{i_1 \ldots i_{m-2l}}.$$

Taking into account that for given N, the number of even members is $\left[\frac{N}{2}\right]$ and odd members – $\left[\frac{N+1}{2}\right]$. Besides this, the even members lead to tensors with even rank and odd members leads to tensor with odd rank. Thus,

$$\sum_{m=0}^{N} \tilde{F}_M n_M = \sum_{s=0}^{\left[\frac{N}{2}\right]} \sum_{l=0}^{\left[\frac{m}{2}\right]} \tilde{a}(2s,l) \tilde{F}_{<i_1 \ldots i_{2s-2l}>k_1k_1 \ldots k_lk_l} n_{i_1 \ldots i_{2s-2l}} + \sum_{s=0}^{\left[\frac{N+1}{2}\right]} \sum_{l=0}^{\left[\frac{m}{2}\right]} \tilde{a}(2s+1,l) \tilde{F}_{<i_1 \ldots i_{2s+1-2l}>k_1k_1 \ldots k_lk_l} n_{i_1 \ldots i_{2s+1-2l}}.$$

For changing of summation order, introduce index $j=s-l$ ($2j=2s-2l$, $2j+1=2s+1-2l$) and collect members with the same rank of tensors:

$$\hat{F}_{2J} = \sum_{s=j}^{\left[\frac{N}{2}\right]} \tilde{a}(2s,s-j) \tilde{F}_{<2J>K_{S-J}K_{S-J}}, \qquad \hat{F}_{2J+1} = \sum_{s=j}^{\left[\frac{N-1}{2}\right]} \tilde{a}(2s+1,s-j) \tilde{F}_{<2J+1>K_{S-J}K_{S-J}}.$$

Then, $\sum_{m=0}^{N} \tilde{F}_M n_M = \sum_{m=0}^{N} \hat{F}_M n_M$. Thus,the proving of (3) is completed. So, theorem is proved.

The first cumulative STF coefficients evaluation

Because the cumulative STF coefficients are used the coefficients of STF part of tensor at first let's to evalute the first of these. According formulae [4]:

$$a(n,l)=\frac{n!}{(2n-4+\dim)!!}\frac{(-1)^l(2n-4+\dim-2l)!!}{(2l)!!(n-2l)!} \quad \text{(S.3)}$$

for given dimension of space dim. Directly from this formulae, one can obtain the expressions for first coefficients which presented in Table S1.

Table S1. The coefficients of STF part of tensor $a(n,l)$ for n from 1 to 7.

n	*L* 0	*1*	*2*	*3*
0	1	–	–	–
1	1	–	–	–
2	1	$-\frac{1}{\dim}$	–	–
3	1	$-\frac{3}{(\dim+2)}$	–	–
4	1	$-\frac{6}{(4+\dim)}$	$\frac{3}{(4+\dim)(2+\dim)}$	–
5	1	$-\frac{10}{(6+\dim)}$	$\frac{15}{(6+\dim)(4+\dim)}$	–
6	1	$-\frac{15}{(8+\dim)}$	$\frac{45}{(8+\dim)(6+\dim)}$	$-\frac{15}{(8+\dim)(6+\dim)(4+\dim)}$
7	1	$-\frac{21}{(10+\dim)}$	$\frac{105}{(10+\dim)(8+\dim)}$	$-\frac{105}{(10+\dim)(8+\dim)(6+\dim)}$

For cumulative STF coefficients evaluation formulas (4)–(6) of theorem were used. The estimation of the cumulative coefficients for *l*=0,1 is trivial: $\tilde{a}(n,0)=1$, $\tilde{a}(n,1)=-a(n,1)$. For *l*=2:

$$\tilde{a}(4,2)=a(4,1)a(2,1)-a(4,2)=\frac{6}{(4+\dim)\dim}-\frac{3}{(4+\dim)(2+\dim)}=$$

$$=\frac{3}{(4+\dim)}\left(\frac{2}{\dim}-\frac{1}{(2+\dim)}\right)=\frac{3}{(2+\dim)\dim};$$

$$\tilde{a}(5,2)=a(5,1)a(3,1)-a(5,2)=\frac{30}{(6+\dim)(2+\dim)}-\frac{15}{(6+\dim)(4+\dim)}=$$

$$=\frac{15}{(6+\dim)}\left(\frac{2}{(2+\dim)}-\frac{1}{(4+\dim)}\right)=\frac{15}{(4+\dim)(2+\dim)};$$

$$\tilde{a}(6,2)=a(6,1)a(4,1)-a(6,2)=\frac{90}{(8+\dim)(4+\dim)}-\frac{45}{(8+\dim)(6+\dim)}=$$

$$=\frac{45}{(8+\dim)}\left(\frac{2}{(4+\dim)}-\frac{1}{(6+\dim)}\right)=\frac{45(8+\dim)}{(8+\dim)(6+\dim)(4+\dim)}=\frac{45}{(6+\dim)(4+\dim)};$$

$$\tilde{a}(7,2)=a(7,1)a(5,1)-a(7,2)=\frac{210}{(10+\dim)(6+\dim)}-\frac{105}{(10+\dim)(8+\dim)}=$$

$$=\frac{105}{(10+\dim)}\left(\frac{2}{(6+\dim)}-\frac{1}{(8+\dim)}\right)=\frac{105(10+\dim)}{(10+\dim)(8+\dim)(6+\dim)}=\frac{105}{(8+\dim)(6+\dim)};$$

For l=3:

$$a(6,1,1)=a(6,1)a(4,2)-a(6,3)=-\frac{45}{(8+\dim)(4+\dim)(2+\dim)}+\frac{15}{(8+\dim)(6+\dim)(4+\dim)}=$$

$$=\frac{15}{(8+\dim)(4+\dim)}\left(\frac{2+\dim}{(6+\dim)}-\frac{18+3\dim}{(2+\dim)}\right)=-\frac{30}{(6+\dim)(4+\dim)(2+\dim)}$$

$$\tilde{a}(6,3)=a(6,0,2)=a(6,1,1)-a(2,1)\tilde{a}(6,2)=-\frac{30}{(6+\dim)(4+\dim)(2+\dim)}+\frac{1}{\dim}\frac{45}{(6+\dim)(4+\dim)}=$$

$$=\frac{15}{(6+\dim)(4+\dim)}\left(\frac{3}{\dim}-\frac{2}{(2+\dim)}\right)=\frac{15}{(4+\dim)(2+\dim)\dim},$$

and for n=7:

$$a(7,1,1)=a(7,1)a(5,2)-a(7,3)=-\frac{21}{(10+\dim)}\frac{15}{(6+\dim)(4+\dim)}+\frac{105}{(10+\dim)(8+\dim)(6+\dim)}=$$

$$=\frac{105(-20-2\dim)}{(10+\dim)(8+\dim)(6+\dim)(4+\dim)}=\frac{-210(10+\dim)}{(10+\dim)(8+\dim)(6+\dim)(4+\dim)}=$$

$$=-\frac{210}{(8+\dim)(6+\dim)(4+\dim)}$$

$$\tilde{a}(7,3)=a(7,0,2)=a(7,1,1)-a(3,1)\tilde{a}(7,2)=-\frac{210}{(8+\dim)(6+\dim)(4+\dim)}+$$

$$+\frac{3}{(\dim+2)}\frac{105}{(8+\dim)(6+\dim)}=\frac{105}{(8+\dim)(6+\dim)}\left(\frac{3}{(\dim+2)}-\frac{2}{(4+\dim)}\right)=$$

$$=\frac{105(8+\dim)}{(8+\dim)(6+\dim)(4+\dim)(\dim+2)}=\frac{105}{(6+\dim)(4+\dim)(\dim+2)}$$