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Research article

A distributed quantile estimation algorithm of heavy-tailed distribution with massive datasets

Xiaoyue Xie[1,2,*] and Jian Shi[1,2]

[1] Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100190, China
[2] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

* **Correspondence:** Email: xiexiaoyue16@mails.ucas.edu.cn; Tel: +8615611535356; Fax: +861082541972.

Supplementary

A. The proof of Theorem 3.1

Proof: According to Theorem 2.1 and Assumption A of Hong et al. [19], the proposed distributed algorithm could converge to the set of KKT points when the following assumption A, B and C were satisfied.

A. there exists a positive constant $L_k > 0$ such that

$$\|\nabla_k g_k(x) - \nabla_k g_k(z)\| \le L_k \|x\text{-}z\|, \quad \forall \ x, z \in \Theta, \quad \forall k.$$

Moreover, Θ is a closed convex set on R^2.

B. For all k, the stepsize ρ_k is chosen large enough such that

B1. The η_k subproblem is strongly convex with the strongly convexity coefficient being $\gamma_k(\rho_k)$;

B2. $\rho_k \gamma_k(\rho_k) > 2L_k^2$, and $\rho_k \ge L_k$.

C. $g(x)$ is lower bounded for all $x \in \Theta$.

Therefore, the proposed algorithm only satisfies the above three assumptions.

Since $g_k(x)$ is a smooth function and $g(x) > 0$, then assumption A and C is satisfied. and we have by the mean value theorem

$$L_k \geq \lambda_{\max}(G_k(\eta)), \ \forall \eta \in \Theta, \tag{A.1}$$

where G_k is the Hessian matrix of g_k and $\lambda_{\max}(G_k(\eta))$ is the maximum eigenvalue of $G_k(\eta)$ for $\eta_k \in \Theta$.

On the other hand, let $f_k(\eta)$ be objective function for the η_k subproblem, if assumption B1 is satisfied, then $f_k(\eta)$ satisfies that $\forall \eta_1, \eta_2 \in \Theta$, and

$$f_k(\eta_2) \geq f_k(\eta_1) + \langle \nabla f_k(\eta_1), \eta_2 - \eta_1 \rangle + \frac{\gamma_k(\rho_k)}{2} \| \eta_2 - \eta_1 \|_2^2,$$

where $\gamma_k(\rho_k)$ is strongly convexity coefficient and

$$f_k(\eta) = g_k(\eta) + \langle y_k, \eta \rangle + \frac{\rho_k}{2} \| \eta - \theta \|_2^2 .$$

By Taylor expansion at η_1, which yields

$$\frac{1}{2}(\eta_2 - \eta_1)^T H_k(\eta_0)(\eta_2 - \eta_1) \geq \frac{\gamma_k(\rho_k)}{2} \| \eta_2 - \eta_1 \|_2^2, \ \exists \eta_0 \in \Theta,$$

where H_k is Hessian matrix of function f_k. By the property of matrix eigenvalues, we have

$$\gamma_k(\rho_k) \leq \lambda_{\min}(H_k(\eta)), \ \forall \eta \in \Theta. \tag{A.2}$$

When assumption B2 is satisfified, by Eq (A-1) and Eq (A.2), then we have

$$\rho_k \lambda_{\min}(H_k(\eta)) > 2\lambda_{\max}^2(G_k(\eta)), \ \forall \eta \in \Theta.$$

And since

$$\lambda_{\min}(H_k(\eta)) = \lambda_{\min}(G_k(\eta)) + \rho_k,$$

then

$$\rho_k(\lambda_{\min}(G_k) + \rho_k) > 2\lambda_{\max}^2(G_k(\eta)), \tag{A.3}$$

where

$$\lambda_{\min}(G_k) = \min_{\eta} \lambda_{\min}(G_k(\eta)),$$
$$\lambda_{\max}(G_k) = \max_{\eta} \lambda_{\max}(G_k(\eta)).$$

Clearly, when the assumption B is satisfied, then the Eq (A.3) is also satisfied, and vice versa.