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Research article

Time-adaptive Lagrangian variational integrators for accelerated optimization

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Appendix

A. Proof of Theorem 3.2

Theorem A.1. *The Type I discrete Hamilton's variational principle,*

$$\delta \bar{\mathfrak{S}}_d \left(\{ (q_k, \mathfrak{q}_k, \lambda_k) \}_{k=0}^N \right) = 0,$$

where,

$$\bar{\mathfrak{S}}_d \left(\{ (q_k, \mathfrak{q}_k, \lambda_k) \}_{k=0}^N \right) = \sum_{k=0}^{N-1} \left[L_d(q_k, \mathfrak{q}_k, q_{k+1}, \mathfrak{q}_{k+1}) - \lambda_k \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} + \lambda_k g(\mathfrak{q}_k) \right] \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k},$$

is equivalent to the discrete extended Euler–Lagrange equations,

$$\mathfrak{q}_{k+1} = \mathfrak{q}_k + (\tau_{k+1} - \tau_k) g(\mathfrak{q}_k),$$

$$\frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} D_1 L_d(q_k, \mathfrak{q}_k, q_{k+1}, \mathfrak{q}_{k+1}) + \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} D_3 L_d(q_{k-1}, \mathfrak{q}_{k-1}, q_k, \mathfrak{q}_k) = 0,$$

$$\begin{aligned} &\left[D_2 L_{d_k} + \lambda_k \frac{1}{\tau_{k+1} - \tau_k} + \lambda_k \nabla g(\mathfrak{q}_k) \right] \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} - \frac{1}{\tau_{k+1} - \tau_k} \left[L_{d_k} - \lambda_k \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} + \lambda_k g(\mathfrak{q}_k) \right] \\ &+ \left[D_4 L_{d_{k-1}} - \lambda_{k-1} \frac{1}{\tau_k - \tau_{k-1}} \right] \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} + \frac{1}{\tau_k - \tau_{k-1}} \left[L_{d_{k-1}} - \lambda_{k-1} \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} + \lambda_{k-1} g(\mathfrak{q}_{k-1}) \right] = 0, \end{aligned}$$

where L_{d_k} denotes $L_d(q_k, \mathfrak{q}_k, q_{k+1}, \mathfrak{q}_{k+1})$.

Proof. We use the notation $L_{d_k} = L_d(q_k, \mathfrak{q}_k, q_{k+1}, \mathfrak{q}_{k+1})$, and we will use the fact that

$$\delta q_0 = \delta q_N = \delta \mathfrak{q}_0 = \delta \mathfrak{q}_N = 0$$

throughout the proof. We have

$$\begin{aligned}
\delta \bar{\mathfrak{S}}_d &= \delta \left(\sum_{k=0}^{N-1} \left[L_d(q_k, \mathfrak{q}_k, q_{k+1}, \mathfrak{q}_{k+1}) - \lambda_k \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} + \lambda_k g(\mathfrak{q}_k) \right] \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \right) \\
&= \sum_{k=1}^{N-1} \left[D_2 L_{d_k} + \lambda_k \frac{1}{\tau_{k+1} - \tau_k} + \lambda_k \nabla g(\mathfrak{q}_k) \right] \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \delta \mathfrak{q}_k \\
&\quad - \sum_{k=1}^{N-1} \frac{1}{\tau_{k+1} - \tau_k} \left[L_{d_k} - \lambda_k \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} + \lambda_k g(\mathfrak{q}_k) \right] \delta \mathfrak{q}_k \\
&\quad + \sum_{k=0}^{N-2} \left[D_4 L_{d_k} - \lambda_k \frac{1}{\tau_{k+1} - \tau_k} \right] \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \delta \mathfrak{q}_{k+1} \\
&\quad + \sum_{k=0}^{N-2} \frac{1}{\tau_{k+1} - \tau_k} \left[L_{d_k} - \lambda_k \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} + \lambda_k g(\mathfrak{q}_k) \right] \delta \mathfrak{q}_{k+1} \\
&\quad + \sum_{k=1}^{N-1} \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} D_1 L_{d_k} \delta q_k \\
&\quad + \sum_{k=0}^{N-2} \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} D_3 L_{d_k} \delta q_{k+1} + \sum_{k=0}^{N-1} \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \left(g(\mathfrak{q}_k) - \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \right) \delta \lambda_k .
\end{aligned}$$

Thus,

$$\begin{aligned}
\delta \bar{\mathfrak{S}}_d = &\sum_{k=1}^{N-1} \left[D_2 L_{d_k} + \lambda_k \frac{1}{\tau_{k+1} - \tau_k} + \lambda_k \nabla g(\mathfrak{q}_k) \right] \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \delta \mathfrak{q}_k \\
&- \sum_{k=1}^{N-1} \frac{1}{\tau_{k+1} - \tau_k} \left[L_{d_k} - \lambda_k \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} + \lambda_k g(\mathfrak{q}_k) \right] \delta \mathfrak{q}_k \\
&+ \sum_{k=1}^{N-1} \left[D_4 L_{d_{k-1}} - \lambda_{k-1} \frac{1}{\tau_k - \tau_{k-1}} \right] \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} \delta \mathfrak{q}_k \\
&+ \sum_{k=1}^{N-1} \frac{1}{\tau_k - \tau_{k-1}} \left[L_{d_{k-1}} - \lambda_{k-1} \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} + \lambda_{k-1} g(\mathfrak{q}_{k-1}) \right] \delta \mathfrak{q}_k \\
&+ \sum_{k=1}^{N-1} \left[\frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} D_1 L_{d_k} + \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} D_3 L_{d_{k-1}} \right] \delta q_k \\
&+ \sum_{k=0}^{N-1} \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \left(g(\mathfrak{q}_k) - \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} \right) \delta \lambda_k .
\end{aligned}$$

As a consequence, if

$$\mathfrak{q}_{k+1} = \mathfrak{q}_k + (\tau_{k+1} - \tau_k) g(\mathfrak{q}_k),$$

$$\frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} D_1 L_d(q_k, \mathfrak{q}_k, q_{k+1}, \mathfrak{q}_{k+1}) + \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} D_3 L_d(q_{k-1}, \mathfrak{q}_{k-1}, q_k, \mathfrak{q}_k) = 0,$$

$$\begin{aligned}
&\left[D_2 L_{d_k} + \lambda_k \frac{1}{\tau_{k+1} - \tau_k} + \lambda_k \nabla g(\mathfrak{q}_k) \right] \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} - \frac{1}{\tau_{k+1} - \tau_k} \left[L_{d_k} - \lambda_k \frac{\mathfrak{q}_{k+1} - \mathfrak{q}_k}{\tau_{k+1} - \tau_k} + \lambda_k g(\mathfrak{q}_k) \right] \\
&\quad + \left[D_4 L_{d_{k-1}} - \lambda_{k-1} \frac{1}{\tau_k - \tau_{k-1}} \right] \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} + \frac{1}{\tau_k - \tau_{k-1}} \left[L_{d_{k-1}} - \lambda_{k-1} \frac{\mathfrak{q}_k - \mathfrak{q}_{k-1}}{\tau_k - \tau_{k-1}} + \lambda_{k-1} g(\mathfrak{q}_{k-1}) \right] = 0,
\end{aligned}$$

then $\delta \bar{\mathfrak{S}}_d \left(\{ (q_k, \mathfrak{q}_k, \lambda_k) \}_{k=0}^{N} \right) = 0$. Conversely, if $\delta \bar{\mathfrak{S}}_d \left(\{ (q_k, \mathfrak{q}_k, \lambda_k) \}_{k=0}^{N} \right) = 0$, then a discrete fundamental theorem of the calculus of variations yields the above equations. □

B. Proof of Theorem 3.4

Theorem B.1. *The Type I discrete Hamilton's variational principle,*

$$\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=0,$$

where,

$$\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=\sum_{k=0}^{N-1}\left\{\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left[L_d(q_k,\mathfrak{q}_k,q_{k+1},\mathfrak{q}_{k+1})-\lambda_k\right]+\lambda_k g(\mathfrak{q}_k)\right\},$$

is equivalent to the discrete extended Euler–Lagrange equations,

$$\mathfrak{q}_{k+1}=\mathfrak{q}_k+(\tau_{k+1}-\tau_k)g(\mathfrak{q}_k),$$

$$\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_1L_d(q_k,\mathfrak{q}_k,q_{k+1},\mathfrak{q}_{k+1})+\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}D_3L_d(q_{k-1},\mathfrak{q}_{k-1},q_k,\mathfrak{q}_k)=0,$$

$$\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_2L_{d_k}-\frac{1}{\tau_{k+1}-\tau_k}L_{d_k}+\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}D_4L_{d_{k-1}}+\frac{1}{\tau_k-\tau_{k-1}}L_{d_{k-1}}=\frac{\lambda_{k-1}}{\tau_k-\tau_{k-1}}-\frac{\lambda_k}{\tau_{k+1}-\tau_k}-\lambda_k\nabla g(\mathfrak{q}_k),$$

where L_{d_k} denotes $L_d(q_k,\mathfrak{q}_k,q_{k+1},\mathfrak{q}_{k+1})$.

Proof. We use the notation $L_{d_k}=L_d(q_k,\mathfrak{q}_k,q_{k+1},\mathfrak{q}_{k+1})$, and we will use the fact that

$$\delta q_0=\delta q_N=\delta\mathfrak{q}_0=\delta\mathfrak{q}_N=0$$

throughout the proof. We have

$$\begin{aligned}
\delta\bar{\mathfrak{S}}_d&=\delta\left(\sum_{k=0}^{N-1}\left\{\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left[L_d(q_k,\mathfrak{q}_k,q_{k+1},\mathfrak{q}_{k+1})-\lambda_k\right]+\lambda_k g(\mathfrak{q}_k)\right\}\right)\\
&=\sum_{k=1}^{N-1}\left[\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_2L_{d_k}-\frac{1}{\tau_{k+1}-\tau_k}L_{d_k}+\frac{\lambda_k}{\tau_{k+1}-\tau_k}+\lambda_k\nabla g(\mathfrak{q}_k)\right]\delta\mathfrak{q}_k\\
&\qquad+\sum_{k=0}^{N-2}\left[\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_4L_{d_k}+\frac{1}{\tau_{k+1}-\tau_k}L_{d_k}-\frac{\lambda_k}{\tau_{k+1}-\tau_k}\right]\delta\mathfrak{q}_{k+1}\\
&\qquad+\sum_{k=1}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_1L_{d_k}\delta q_k+\sum_{k=0}^{N-2}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_3L_{d_k}\delta q_{k+1}\\
&\qquad+\sum_{k=0}^{N-1}\left(g(\mathfrak{q}_k)-\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\right)\delta\lambda_k.
\end{aligned}$$

Thus,

$$\begin{aligned}
\delta\bar{\mathfrak{S}}_d=&\sum_{k=1}^{N-1}\left[\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_2L_{d_k}-\frac{1}{\tau_{k+1}-\tau_k}L_{d_k}+\frac{\lambda_k}{\tau_{k+1}-\tau_k}+\lambda_k\nabla g(\mathfrak{q}_k)\right]\delta\mathfrak{q}_k\\
&+\sum_{k=1}^{N-1}\left[\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}D_4L_{d_{k-1}}+\frac{1}{\tau_k-\tau_{k-1}}L_{d_{k-1}}-\frac{\lambda_{k-1}}{\tau_k-\tau_{k-1}}\right]\delta\mathfrak{q}_k\\
&+\sum_{k=1}^{N-1}\left[\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_1L_{d_k}+\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}D_3L_{d_{k-1}}\right]\delta q_k+\sum_{k=0}^{N-1}\left(g(\mathfrak{q}_k)-\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\right)\delta\lambda_k.
\end{aligned}$$

As a consequence, if

$$\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_2L_{d_k}-\frac{1}{\tau_{k+1}-\tau_k}L_{d_k}+\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}D_4L_{d_{k-1}}+\frac{1}{\tau_k-\tau_{k-1}}L_{d_{k-1}}=\frac{\lambda_{k-1}}{\tau_k-\tau_{k-1}}-\frac{\lambda_k}{\tau_{k+1}-\tau_k}-\lambda_k\nabla g(\mathfrak{q}_k),$$

$$\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_1L_d(q_k,\mathfrak{q}_k,q_{k+1},\mathfrak{q}_{k+1})+\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}D_3L_d(q_{k-1},\mathfrak{q}_{k-1},q_k,\mathfrak{q}_k)=0,$$

$$\mathfrak{q}_{k+1}=\mathfrak{q}_k+(\tau_{k+1}-\tau_k)g(\mathfrak{q}_k),$$

then $\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=0$.

Conversely, if $\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=0$, then a discrete fundamental theorem of the calculus of variations yields the above equations. □

C. Proof of Theorem 5.1

Theorem C.1. *The Type I discrete Hamilton's variational principle,*

$$\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=0,$$

where,

$$\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=\sum_{k=0}^{N-1}\left[L_d(q_k,f_k,\mathfrak{q}_k,\mathfrak{q}_{k+1})-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_kg(\mathfrak{q}_k)\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k},$$

is equivalent to the discrete extended Euler–Lagrange equations,

$$\mathfrak{q}_{k+1}=\mathfrak{q}_k+(\tau_{k+1}-\tau_k)g(\mathfrak{q}_k),$$

$$\mathrm{Ad}^*_{f_k^{-1}}\left(\mathrm{T}_e^*\mathrm{L}_{f_k}D_2L_{d_k}\right)=\mathrm{T}_e^*\mathrm{L}_{q_k}D_1L_{d_k}+\frac{\tau_{k+1}-\tau_k}{\mathfrak{q}_{k+1}-\mathfrak{q}_k}\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}\mathrm{T}_e^*\mathrm{L}_{f_{k-1}}D_2L_{d_{k-1}},$$

$$\begin{aligned}&\left[D_3L_{d_k}+\lambda_k\frac{1}{\tau_{k+1}-\tau_k}+\lambda_k\nabla g(\mathfrak{q}_k)\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}-\frac{1}{\tau_{k+1}-\tau_k}\left[L_{d_k}-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_kg(\mathfrak{q}_k)\right]\\&+\left[D_4L_{d_k}-\lambda_{k-1}\frac{1}{\tau_k-\tau_{k-1}}\right]\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}+\frac{1}{\tau_k-\tau_{k-1}}\left[L_{d_{k-1}}-\lambda_{k-1}\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}+\lambda_{k-1}g(\mathfrak{q}_{k-1})\right]=0,\end{aligned}$$

where L_{d_k} denotes $L_d(q_k,f_k,\mathfrak{q}_k,\mathfrak{q}_{k+1})$.

Proof. We will use the notation $L_{d_k}=L_d(q_k,f_k,\mathfrak{q}_k,\mathfrak{q}_{k+1})$, and we will use the boundary conditions

$$\delta q_0=\delta q_N=\delta\mathfrak{q}_0=\delta\mathfrak{q}_N=\eta_0=\eta_N=0$$

throughout the proof.

We have

$$\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=\delta\left(\sum_{k=0}^{N-1}\left[L_d(q_k,f_k,\mathfrak{q}_k,\mathfrak{q}_{k+1})-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_k g(\mathfrak{q}_k)\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\right)$$
$$=\sum_{k=1}^{N-1}\left[D_3L_{d_k}+\lambda_k\frac{1}{\tau_{k+1}-\tau_k}+\lambda_k\nabla g(\mathfrak{q}_k)\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\delta\mathfrak{q}_k$$
$$-\sum_{k=1}^{N-1}\frac{1}{\tau_{k+1}-\tau_k}\left[L_{d_k}-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_k g(\mathfrak{q}_k)\right]\delta\mathfrak{q}_k$$
$$+\sum_{k=0}^{N-2}\left[D_4L_{d_k}-\lambda_k\frac{1}{\tau_{k+1}-\tau_k}\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\delta\mathfrak{q}_{k+1}$$
$$+\sum_{k=0}^{N-2}\frac{1}{\tau_{k+1}-\tau_k}\left[L_{d_k}-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_k g(\mathfrak{q}_k)\right]\delta\mathfrak{q}_{k+1}$$
$$+\sum_{k=1}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_1L_{d_k}\delta q_k$$
$$+\sum_{k=0}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}D_2L_{d_k}\delta f_k$$
$$+\sum_{k=0}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left(g(\mathfrak{q}_k)-\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\right)\delta\lambda_k.$$

We can write δg_k as $\delta g_k = g_k\eta_k$ for some $\eta_k\in\mathfrak{g}$. Then, taking the variation of the discrete kinematics equation $q_{k+1}=q_kf_k$ gives the equation $\delta q_{k+1}=\delta q_kf_k+q_k\delta f_k$ and $f_k=q_k^{-1}q_{k+1}$. As a consequence,

$$\delta f_k=q_k^{-1}\delta q_{k+1}-q_k^{-1}\delta q_kf_k=q_k^{-1}q_{k+1}\eta_{k+1}-q_k^{-1}q_k\eta_kf_k=f_k\eta_{k+1}-\eta_kf_k,$$

so

$$\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=\sum_{k=1}^{N-1}\left[D_3L_{d_k}+\lambda_k\frac{1}{\tau_{k+1}-\tau_k}+\lambda_k\nabla g(\mathfrak{q}_k)\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\delta\mathfrak{q}_k$$
$$-\sum_{k=1}^{N-1}\frac{1}{\tau_{k+1}-\tau_k}\left[L_{d_k}-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_k g(\mathfrak{q}_k)\right]\delta\mathfrak{q}_k$$
$$+\sum_{k=1}^{N-1}\left(D_4L_{d_{k-1}}-\lambda_{k-1}\frac{1}{\tau_k-\tau_{k-1}}\right)\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}\delta\mathfrak{q}_k$$
$$+\sum_{k=1}^{N-1}\frac{1}{\tau_k-\tau_{k-1}}\left(L_{d_{k-1}}-\lambda_{k-1}\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}+\lambda_{k-1}g(\mathfrak{q}_{k-1})\right)\delta\mathfrak{q}_k$$
$$+\sum_{k=1}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left(\mathrm{T}_e^*\mathrm{L}_{q_k}D_1L_{d_k}\bullet\eta_k\right)$$
$$+\sum_{k=0}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left(\mathrm{T}_e^*\mathrm{L}_{f_k}D_2L_{d_k}\bullet\left[\eta_{k+1}-f_k^{-1}\eta_kf_k\right]\right)$$
$$+\sum_{k=0}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left(g(\mathfrak{q}_k)-\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\right)\delta\lambda_k.$$

Then,

$$
\begin{aligned}
\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right) = &\sum_{k=1}^{N-1}\left[D_3L_{d_k}+\lambda_k\frac{1}{\tau_{k+1}-\tau_k}+\lambda_k\nabla g(\mathfrak{q}_k)\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\delta\mathfrak{q}_k\\
&-\sum_{k=1}^{N-1}\frac{1}{\tau_{k+1}-\tau_k}\left[L_{d_k}-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_k g(\mathfrak{q}_k)\right]\delta\mathfrak{q}_k\\
&+\sum_{k=1}^{N-1}\left(D_4L_{d_{k-1}}-\lambda_{k-1}\frac{1}{\tau_k-\tau_{k-1}}\right)\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}\delta\mathfrak{q}_k\\
&+\sum_{k=1}^{N-1}\frac{1}{\tau_k-\tau_{k-1}}\left(L_{d_{k-1}}-\lambda_{k-1}\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}+\lambda_{k-1}g(\mathfrak{q}_{k-1})\right)\delta\mathfrak{q}_k\\
&+\sum_{k=1}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left(\mathrm{T}_e^*\mathrm{L}_{q_k}D_1L_{d_k}\bullet\eta_k\right)\\
&+\sum_{k=0}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left(g(\mathfrak{q}_k)-\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\right)\delta\lambda_k\\
&+\sum_{k=0}^{N-1}\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}\left(\mathrm{T}_e^*\mathrm{L}_{f_{k-1}}D_2L_{d_{k-1}}\bullet\eta_k\right)\\
&-\sum_{k=0}^{N-1}\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}\left(\mathrm{T}_e^*\mathrm{L}_{f_k}D_2L_{d_k}\bullet\mathrm{Ad}_{f_k^{-1}}\eta_k\right).
\end{aligned}
$$

As a consequence, if

$$\mathfrak{q}_{k+1}=\mathfrak{q}_k+(\tau_{k+1}-\tau_k)g(\mathfrak{q}_k),$$

$$\mathrm{Ad}^*_{f_k^{-1}}\left(\mathrm{T}_e^*\mathrm{L}_{f_k}D_2L_{d_k}\right)=\mathrm{T}_e^*\mathrm{L}_{q_k}D_1L_{d_k}+\frac{\tau_{k+1}-\tau_k}{\mathfrak{q}_{k+1}-\mathfrak{q}_k}\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}\mathrm{T}_e^*\mathrm{L}_{f_{k-1}}D_2L_{d_{k-1}},$$

$$
\begin{aligned}
&\left[D_3L_{d_k}+\lambda_k\frac{1}{\tau_{k+1}-\tau_k}+\lambda_k\nabla g(\mathfrak{q}_k)\right]\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}-\frac{1}{\tau_{k+1}-\tau_k}\left[L_{d_k}-\lambda_k\frac{\mathfrak{q}_{k+1}-\mathfrak{q}_k}{\tau_{k+1}-\tau_k}+\lambda_k g(\mathfrak{q}_k)\right]\\
&\quad+\left[D_4L_{d_k}-\lambda_{k-1}\frac{1}{\tau_k-\tau_{k-1}}\right]\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}+\frac{1}{\tau_k-\tau_{k-1}}\left[L_{d_{k-1}}-\lambda_{k-1}\frac{\mathfrak{q}_k-\mathfrak{q}_{k-1}}{\tau_k-\tau_{k-1}}+\lambda_{k-1}g(\mathfrak{q}_{k-1})\right]=0,
\end{aligned}
$$

then $\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=0$. Conversely, if $\delta\bar{\mathfrak{S}}_d\left(\{(q_k,\mathfrak{q}_k,\lambda_k)\}_{k=0}^N\right)=0$, then a discrete fundamental theorem of the calculus of variations yields the above equations. □