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Research article

Proving prediction prudence

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A. Appendix: Special cases of the weighted paired difference approach

Equal weights in the basic approach. In this case, the variable of interest is the ordinary average of the sample $\Delta_1, \ldots, \Delta_n$, as reflected by the fact that then instead of (2.4a), it holds that

$$\mathrm{E}[X_\vartheta] \;=\; \frac{1}{n}\sum_{i=1}^{n} \Delta_i - \vartheta. \tag{A.1}$$

In the same vein, the algorithms and formulae of Section 2.1 can be adapted to the equal weights case by replacing all weights w_i and w_j with $1/n$.

Weight-adjusted sample. In this case, the weights w_i are accounted for by replacing the sample $\Delta_1, \ldots, \Delta_n$ with the sample $\Delta_1^*, \ldots, \Delta_n^*$ where Δ_i^* is defined by

$$\Delta_i^* \;=\; w_i\, \Delta_i.$$

The adjusted sample $\Delta_1^*, \ldots, \Delta_n^*$ in turn is treated as in the equal weights case. Then, in particular, (2.3) for the distribution of X_ϑ reads

$$\mathrm{P}[X_\vartheta = \Delta_i^* - \vartheta] \;=\; \frac{1}{n}, \qquad i = 1, \ldots, n.$$

If $\sum_{i=1}^{n} w_i\, \Delta_i \neq 0$, it follows that

$$\mathrm{E}[X_\vartheta] \;=\; \frac{1}{n}\sum_{i=1}^{n} \Delta_i^* - \vartheta \;\neq\; \sum_{i=1}^{n} w_i\, \Delta_i - \vartheta. \tag{A.2}$$

As a consequence of (A.2), the adaptation of the algorithms and formulae from Section (2.1) for the weight-adjusted sample case would appear somewhat misleading if comparability in magnitude of the values of the test statistic $\bar{X}_\vartheta$ to its values in the unequal weights case as discussed in Section 2.1 were intended.

A workaround for this problem is to adjust the sample not only for the weights but also for the sample size, i.e. to define the adjusted sample $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_n$ by

$$\widetilde{\Delta}_i \;=\; n\, w_i\, \Delta_i. \tag{A.3a}$$

Assuming equal weights now means $\mathrm{P}[X_\vartheta = \widetilde{\Delta}_i - \vartheta] = 1/n$ which implies

$$\begin{aligned}
\mathrm{E}[X_\vartheta] &= \frac{1}{n}\sum_{i=1}^{n} \widetilde{\Delta}_i - \vartheta \\
&= \sum_{i=1}^{n} w_i\, \Delta_i - \vartheta, && \text{(A.3b)}\\
\mathrm{var}[X_\vartheta] &= \frac{1}{n}\sum_{i=1}^{n} \widetilde{\Delta}_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} \widetilde{\Delta}\right)^2 \\
&= n \sum_{i=1}^{n} w_i^2\, \Delta_i^2 - \left(\sum_{i=1}^{n} w_i\, \Delta_i\right)^2. && \text{(A.3c)}
\end{aligned}$$

Comparison with (2.4b) shows that the variances of X_ϑ according to the weighting scheme (A.3a) and the weighting scheme deployed in Section 2.1 differ by

$$\sum_{i=1}^{n} (n\, w_i - 1)\, w_i\, \Delta_i^2,$$

which can be positive or negative. The algorithms and formulae from Section 2.1 can be applied to the weight-adjusted sample case as specified by (A.3a) and $\mathrm{P}[X_\vartheta = \widetilde{\Delta}_i - \vartheta] = 1/n$ if the following two modifications are taken into account in the given order:

- Replace the value of Δ_i by the value of $\widetilde{\Delta}_i \;=\; n\, w_i\, \Delta_i$ for $i = 1, \ldots, n$.
- Replace all remaining appearances of the weights w_i by $1/n$.

Note that the weight-adjustment (A.3a) can also be deployed for samples with more special structure like the ones considered in Section (2.3) and Appendix B below. There is no guarantee, however, that adjustment (A.3a) would preserve the 'values in the unit interval' constraint of Section (2.3). There is no such preservation issue with regard to Appendix B.

B. Appendix: Tests for non-negative variables

In contrast to LGD and CCF which by definition are variables with values in the unit interval, EAD in principle may take any non-negative value. This requires some modifications in order to adapt the approach from Section 2.3 to the assessment of EAD estimates.

Starting point.

- A sample of paired observations $(h_1, \eta_1), \ldots, (h_n, \eta_n)$, with predicted EADs $0 < \eta_i < \infty$ and realised exposures $0 \le h_i < \infty$.
- Weights $0 < w_i < 1$, $i = 1, \ldots, n$, with $\sum_{i=1}^n w_i = 1$,
- Weighted average observed EAD $h_w = \sum_{i=1}^n w_i h_i$ and weighted average EAD prediction $\eta_w = \sum_{i=1}^n w_i \eta_i$.

Interpretation in the context of EAD back-testing.

- A sample of n defaulted credit facilities / loans is analysed.
- The EAD η_i is an estimate of loan i's exposure at the moment of the default, measured in currency units.
- The realized exposure h_i shows the loan i's exposure at the time of default.
- The weight w_i reflects the relative importance of observation i. In the case of direct EAD predictions, one might choose w_i according to (2.2a).
- Define $\Delta_i = h_i - \eta_i$, $i = 1, \ldots, n$. If $|\Delta_i| \approx 0$ then η_i is a good EAD prediction. If $|\Delta_i|$ is large then η_i is a poor EAD prediction.

Goal. We want to use the observed weighted average difference / residual $\Delta_w = \sum_{i=1}^n w_i \Delta_i = h_w - \eta_w$ to assess the quality of the calibration of the model / approach for the η_i to predict the realised exposures h_i. Again we want to answer the following two questions:

- If $\Delta_w < 0$, how safe is the conclusion that the observed (realised) values are on weighted average less than the predictions, i.e. the predictions are prudent / conservative?
- If $\Delta_w > 0$, how safe is the conclusion that the observed (realised) values are on weighted average greater than the predictions, i.e. the predictions are aggressive?

The safety of such conclusions is measured by p-values which provide error probabilities for the conclusions to be wrong. The lower the p-value, the more likely the conclusion is right.

In order to be able to examine the specific properties of the sample and Δ_w with statistical methods, we have to make the assumption that the sample was generated with some random mechanism. This mechanism is described in the following modification of Assumption 2.4.

Assumption B.1. *The sample $\Delta_1, \ldots, \Delta_n$ consists of independent realisations of a random variable X_ϑ with distribution given by*

$$X_\vartheta = h_I - Y_\vartheta, \tag{B.1a}$$

where I is a random variable with values in $\{1, \ldots, n\}$ and $\mathrm{P}[I = i] = w_i$, $i = 1, \ldots, n$. Y_ϑ is a gamma(α_i,β_i)-distributed random variable conditional on $I = i$ for $i = 1, \ldots, n$. The parameters α_i and β_i of the gamma-distribution depend on the unknown parameter $0 < \vartheta < \infty$ by*

$$\begin{aligned} \alpha_i &= \frac{\vartheta_i}{v}, \qquad \textit{and} \\ \beta_i &= v. \end{aligned} \tag{B.1b}$$

*See Casella and Berger [2002, Section 3.3] for a definition of the gamma-distribution.

In (B.1b)*, the constant* $0 < v < \infty$ *is the same for all i. The* ϑ_i *are determined by*

$$\vartheta_i \;=\; \eta_i\,\frac{\vartheta}{\eta_w}. \tag{B.1c}$$

Note that Assumption B.1 describes a method for recalibration of the EAD estimates $\eta_1, \ldots, \eta_n$ to match targets ϑ with the weighted average of the ϑ_i. By definition of Y_ϑ, it holds that $\mathrm{E}[Y_\vartheta \,|\, I = i] = \vartheta_i$.

The constant v specifies the variance of Y_ϑ conditional on $I = i$ as multiple of its expected value ϑ_i, i.e. it holds that

$$\mathrm{var}[Y_\vartheta \,|\, I = i] \;=\; v\,\vartheta_i, \qquad i = 1, \ldots, n. \tag{B.2}$$

The constant v must be pre-defined or separately estimated. We suggest estimating it from the sample $h_1, \ldots, h_n$ as

$$\hat{v} \;=\; \frac{\sum_{i=1}^n w_i\,h_i^2 - h_w^2}{h_w}. \tag{B.3}$$

Proposition B.2. *For* X_ϑ *as described in Assumption B.1, the expected value and the variance are given by*

$$\mathrm{E}[X_\vartheta] = h_w - \vartheta, \text{ and} \tag{B.4a}$$

$$\mathrm{var}[X_\vartheta] = \sum_{i=1}^n w_i\,(h_i - \vartheta_i)^2 - (h_w - \vartheta)^2 + v\,\vartheta. \tag{B.4b}$$

Proof. For deriving the formula for $\mathrm{var}[X_\vartheta]$, make use of the well-known variance decomposition

$$\mathrm{var}[X_\vartheta] = \mathrm{E}[\mathrm{var}[X_\vartheta \,|\, I]] + \mathrm{var}[\mathrm{E}[X_\vartheta \,|\, I]].$$

Like in (2.13b), the variance of X_ϑ as shown in (B.4b) depends on the parameter ϑ and has an additional component $v\,\vartheta$ which reflects the potentially different variances of the exposures at default in an inhomogeneous portfolio.

By Assumption B.1 and Proposition B.2, the questions on the safety of conclusions from the sign of Δ_w again can be translated into hypotheses on the value of the parameter ϑ:

- If $\Delta_w < 0$, can we conclude that $H_0 : \vartheta \le h_w$ is false and $H_1 : \vartheta > h_w \Leftrightarrow \mathrm{E}[X_\vartheta] < 0$ is true?
- If $\Delta_w > 0$, can we conclude that $H_0^* : \vartheta \ge h_w$ is false and $H_1^* : \vartheta < h_w \Leftrightarrow \mathrm{E}[X_\vartheta] > 0$ is true?

If we assume that the sample $\Delta_1, \ldots, \Delta_n$ was generated by independent realisations of X_ϑ then the distribution of the sample mean is different from the distribution of X_ϑ, as shown in the following corollary to Proposition B.2.

Corollary B.3. *Let* $X_{1,\vartheta}, \ldots, X_{n,\vartheta}$ *be independent and identically distributed copies of* X_ϑ *as in Assumption B.1 and define* $\bar{X}_\vartheta = \frac{1}{n}\sum_{i=1}^n X_{i,\vartheta}$*. Then for the mean and variance of* $\bar{X}_\vartheta$*, it holds that*

$$\mathrm{E}[\bar{X}_\vartheta] = h_w - \vartheta. \tag{B.5a}$$

$$\mathrm{var}[\bar{X}_\vartheta] = \frac{1}{n}\left(\sum_{i=1}^n w_i\,(h_i - \vartheta_i)^2 - (h_w - \vartheta)^2 + v\,\vartheta\right). \tag{B.5b}$$

In the following, we use $\bar{X}_\vartheta$ as the test statistic and interpret $\Delta_w = h_w - \eta_w$ as its observed value.

Proposition B.4. *In the setting of Assumption B.1 and Corollary B.3,* $\vartheta \le \widehat{\vartheta}$ *implies that*

$$\mathrm{P}[\bar{X}_\vartheta \le x] \;\le\; \mathrm{P}[\bar{X}_{\widehat{\vartheta}} \le x], \qquad \text{for all } x \in \mathbb{R}.$$

Proof. Same as the proof of Proposition 2.7.

Bootstrap test. Generate a Monte Carlo sample $\bar{x}_1, \ldots, \bar{x}_R$ from X_ϑ with $\vartheta = h_w$ as follows:

- For $j = 1, \ldots, R$: $\bar{x}_j$ is the equally weighted mean of n independent draws from the distribution of X_ϑ as given by Assumption B.1, with $\vartheta = h_w$.
- $\bar{x}_1, \ldots, \bar{x}_R$ are realisations of independent, identically distributed random variables,

Then a bootstrap p-value for the test of $H_0 : \vartheta \leq h_w$ against $H_1 : \vartheta > h_w$ can be calculated as

$$\text{p-value} \;=\; \frac{1 + \#\{i : i \in \{1, \ldots, n\}, \bar{x}_i \leq h_w - \eta_w\}}{R + 1}. \tag{B.6a}$$

A bootstrap p-value for the test of $H_0^* : \vartheta \geq h_w$ against $H_1^* : \vartheta < h_w$ is given by

$$\text{p-value}^* \;=\; \frac{1 + \#\{i : i \in \{1, \ldots, n\}, \bar{x}_i \geq h_w - \eta_w\}}{R + 1}. \tag{B.6b}$$

Rationale. Same as the rationale for (2.16a) and (2.16b).

Normal approximate test. By Corollary B.3, we find that the distribution of $\bar{X}_{h_w}$ can be approximated by a normal distribution with mean 0 and variance as shown on the right-hand side of (B.5b) with $\vartheta = h_w$. With $x = h_w - \eta_w$, one obtains for the approximate p-value of $H_0 : \vartheta \leq h_w$ against $H_1 : \vartheta > h_w$:

$$\begin{aligned}\text{p-value} \;&=\; \mathrm{P}[\bar{X}_{h_w} \leq x] \\ &\approx\; \Phi\left(\frac{\sqrt{n}\,(h_w - \eta_w)}{\sqrt{\sum_{i=1}^n w_i\,(h_i - \widehat{\vartheta}_i)^2 + v\,h_w}}\right),\end{aligned} \tag{B.7a}$$

with $\widehat{\vartheta}_i = \eta_i \frac{h_w}{\eta_w}$ as in Assumption B.1. The same reasoning gives for the normal approximate p-value of $H_0^* : \vartheta \geq h_w$ against $H_1^* : \vartheta < h_w$:

$$\text{p-value}^* \;\approx\; 1 - \Phi\left(\frac{\sqrt{n}\,(h_w - \eta_w)}{\sqrt{\sum_{i=1}^n w_i\,(h_i - \widehat{\vartheta}_i)^2 + v\,h_w}}\right). \tag{B.7b}$$