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Research article

Brezis Nirenberg type results for local non-local problems under mixed boundary conditions

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Appendix

Proposition 0.1. *The space* $(\mathcal{X}^{1,2}_{\Pi_1}(U), \langle ., . \rangle)$ *is a Hilbert space with scalar product given by*

$$\langle u, v \rangle := \int_O \nabla u . \nabla v \, dx + \iint_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dxdy.$$

Proof. It is easy to check that $\eta(.)$ is a norm on $\mathcal{X}^{1,2}_{\Pi_1}(U)$, since $\eta(u) = 0$ implies $u = 0$ a.e. in $\mathbb{R}^n$ follows straightaway from $\|u\|_{L^2(O)} \leq \zeta(u), \ \forall \ u \in \mathcal{X}^{1,2}_{\Pi_1}(U)$. In order to show that $\mathcal{X}^{1,2}_{\Pi_1}(U)$ is a Hilbert space, we need to prove that $\mathcal{X}^{1,2}_{\Pi_1}(U)$ is complete with respect to the norm $\eta(.)$. For this, let $\{u_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{X}^{1,2}_{\Pi_1}(U)$. From $\|u\|_{L^2(O)} \leq \zeta(u), \ \forall \ u \in \mathcal{X}^{1,2}_{\Pi_1}(U)$, we can easily deduce that $\{u_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^{2^*}(O)$, and since it is a complete Banach space, there exists a $u \in L^{2^*}(O)$ such that $u_j \to u$ in $L^{2^*}(O)$ as $j \to \infty$. Hence, up to a subsequence still denoted by itself such that $u_j \to u$ a.e. in O, for this, we refer [[1], Theorem IV.9]. Clearly, we also have that $\{\nabla u_j\}_j$ is a Cauchy sequence in $L^2(O)$, and hence there exists $w \in L^2(O)$ such that $\nabla u_j \to w$ in $L^2(O)$ as $j \to \infty$. Now we will show that $\nabla u = w$. If we fix $\phi \in C^\infty_0(O)$, then by the definition of weak derivative one has

$$\int_O \frac{\partial u_j}{\partial x_i} \phi \, \mathrm{d}x = - \int_O u_j \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x, \ \forall \ 1 \leq i \leq n. \tag{0.1}$$

Using the fact that strong convergence in $L^{2^*}(O)$ as well as in $L^2(O)$ implies weak convergence in these spaces, we have

$$\int_O u_j \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x \to \int_O u \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x \text{ and } \int_O \frac{\partial u_j}{\partial x_i} \phi \, \mathrm{d}x \to \int_O w_i \phi \, \mathrm{d}x \text{ as } j \to \infty. \tag{0.2}$$

Letting $j \to \infty$ in (0.1) and using (0.2), we obtain

$$\int_O w_i \phi \, \mathrm{d}x = - \int_O u \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x, \ \forall \ 1 \leq i \leq n.$$

It follows at once that
$$\frac{\partial u}{\partial x_i} = w_i \in L^2(O),\ \forall\ 1 \le i \le n,\ i.e.,\ \nabla u = w.$$
Hence, the proof of our claim is finished. Next, we aim to prove that $\mathcal{X}^{1,2}_{\Pi_1}(U)$ is complete. For this, one can notice that $u_j \to u$ a.e. in O as $j \to \infty$. More precisely, it means that there exists a set $D_1 \subset \mathbb{R}^n$ such that
$$|D_1| = 0 \quad \text{and} \quad u_j(x) \to u(x) \text{ as } j \to \infty \quad \text{for all } x \in O \backslash D_1. \tag{0.3}$$
Furthermore, given any $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$, for any $(x, y) \in \mathbb{R}^{2n}$, we consider the following function
$$G_{\mathcal{H}}(x, y) = \left[\frac{(\mathcal{H}(x) - \mathcal{H}(y))\chi_Q(x, y)}{|x - y|^{\frac{n+2s}{2}}}\right]. \tag{0.4}$$
Now, since
$$G_{u_j}(x, y) - G_{u_k}(x, y) = \left[\frac{\left(u_j(x) - u_k(y) - u_j(x) + u_k(y)\right)\chi_Q(x, y)}{|x - y|^{\frac{n+2s}{2}}}\right]$$
and $\{u_j\}_j$ is a Cauchy sequence, we have for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, if $j, k \ge n_\epsilon$, then
$$\epsilon^2 \ge C_{n,s} \iint_Q \frac{\left|\left(u_j - u_k\right)(x) - \left(u_j - u_k\right)(y)\right|^2}{|x - y|^{n+2s}} dxdy = \left\|G_{u_j} - G_{u_k}\right\|^2_{L^2\left(\mathbb{R}^{2n}\right)}.$$
It follows that $\{G_{u_j}\}_j$ is a Cauchy sequence in $L^2\left(\mathbb{R}^{2n}\right)$. From this, we infer there exists $G \in L^2\left(\mathbb{R}^{2n}\right)$ such that $G_{u_j} \to G$ in $L^2\left(\mathbb{R}^{2n}\right)$ as $j \to \infty$, and hence, without loss of generality, we have $G_{u_j} \to G$ a.e. in $\mathbb{R}^{2n}$ as $j \to \infty$. It means that we can find $D_2 \subset \mathbb{R}^{2n}$ such that
$$|D_2| = 0 \text{ and } G_{u_j}(x, y) \to G(x, y) \text{ as } j \to \infty,\ \forall\ (x, y) \in \mathbb{R}^{2n} \backslash D_2. \tag{0.5}$$
For any $x \in O$, we define the following sets such as
$$M_x = \left\{y \in \mathbb{R}^n : (x, y) \in \mathbb{R}^{2n} \backslash Z_2\right\}, \quad P = \{x \in \Omega : |\mathbb{R}^n \backslash M_x| = 0\}$$
and
$$N = \left\{(x, y) \in \mathbb{R}^{2n} : x \in O \text{ and } y \in \mathbb{R}^n \backslash M_x\right\}.$$
Our next goal is to show
$$N \subseteq D_2 \tag{0.6}$$
Indeed, if $(x, y) \in N$, then $y \in \mathbb{R}^n \backslash M_x$, namely $(x, y) \notin \mathbb{R}^{2n} \backslash D_2$, and hence $(x, y) \in D_2$, as desired. In addition, by (0.5) and (0.6), we find that $|N| = 0$. Hence, by Fubini's theorem, it follows that
$$0 = |N| = \int_O |\mathbb{R}^n \backslash M_x|\, dx$$
and thus $|\mathbb{R}^n \backslash M_x| = 0$ for a.e. $x \in O$. Also, we have $|O \backslash P| = 0$ which, together with (0.3), gives
$$|O \backslash (P \backslash D_1)| = |(O \backslash P) \cup D_1| \leqslant |O \backslash P| + |D_1| = 0.$$

In particular, we infer that $P\backslash D_1$ is non-empty. Let us fix $x_0 \in P\backslash D_1$. Now, since $x_0 \in O\backslash D_1$, we have

$$\lim_{j\to+\infty} u_j(x_0) = u(x_0)$$

by (0.3). Moreover, $\left|\mathbb{R}^n\backslash M_{x_0}\right| = 0$, since $x_0 \in P$, namely for any $y \in M_{x_0}$, it follows that $(x_0, y) \in \mathbb{R}^{2n}\backslash D_2$. Hence, by using (0.4) and (0.5), we obtain that

$$\lim_{j\to+\infty} G_{u_j}(x_0, y) = |x_0 - y|^{\frac{-(n+2s)}{2}} \lim_{j\to+\infty} \left(u_j(x_0) - u_k(y)\right)\chi_Q(x_0, y) = G(x_0, y)$$

In addition, since $O \times \left(\mathbb{R}^N\backslash O\right) \subseteq Q$, by the definition in (0.4),

$$G_{u_j}(x_0, y) = \left[\frac{u_j(x_0) - u_k(y)}{|x_0 - y|^{\frac{n+2s}{2}}}\right] \quad \text{for a.e. } y \in \mathbb{R}^n\backslash O.$$

Hence, we have

$$\begin{aligned}\lim_{j\to+\infty} u_j(y) &= \lim_{j\to+\infty}\left(u_k(x_0) - |x_0 - y|^{\frac{n+2s}{2}} G_{u_j}(x_0, y)\right)\\ &= u(x_0) - |x_0 - y|^{\frac{n+2s}{2}} G(x_0, y) = u(y).\end{aligned}$$

This implies that $u_j \to u$ a.e. in $\mathbb{R}^n\backslash O$ as $j \to \infty$. Consequently, by Fatou's lemma, we obtain

$$\begin{aligned}\iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy &\le \liminf_{j\to\infty} \iint_Q \frac{|u_j(x) - u_j(y)|^2}{|x - y|^{n+2s}} dxdy\\ &\liminf_{j\to\infty} \iint_Q \frac{|u_j(x) - u_j(y)|^2}{|x - y|^{n+2s}} dxdy + \liminf_{j\to\infty} \int_O |\nabla u_j|^2 dxdy\\ &= \liminf_{j\to\infty} \eta(u_j)^2 < +\infty.\end{aligned}$$

Hence, we deduce that $[u]_s^2 < +\infty$. Now it remains to show that $\eta(u_j) \to \eta(u)$ as $j \to \infty$. For this, let us take $i \ge n_\epsilon$, then by using Fatou's lemma, we get

$$\begin{aligned}[u_i - u]_s^2 &\le \liminf_{j\to\infty} [u_i - u_j]_s^2\\ &\le \liminf_{j\to\infty} [u_i - u_j]_s^2 + \liminf_{j\to\infty} \|\nabla u_i - \nabla u_j\|^2_{L^2(\mathbb{R}^n)}\\ &\le \liminf_{j\to\infty} \eta(u_i - u_j)^2 \le \epsilon.\end{aligned}$$

Hence, $u_i \to u \in \mathcal{X}^{1,2}_{\Pi_1}(U)$ as $i \to \infty$, which completes the proof.

References

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