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Research article

Stokes-Dirac structures for distributed parameter port-Hamiltonian systems: An analytical viewpoint

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A. Backgrounds on boundary control systems

Let us start with the definition of a boundary control system, as given in [49, Chapter 10].

Definition 1 (Boundary control systems). *Let $\mathcal{Z}, \mathcal{X}, \mathcal{U}$ be three complex Hilbert spaces such that $\mathcal{Z} \subset \mathcal{X}$ with continuous embedding.*

Let $J \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ and $G \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ be two linear operators.

The couple (J, G) is a boundary control system on $(\mathcal{Z}, \mathcal{X}, \mathcal{U})$ if the following holds:

(i) G *is onto;*
(ii) $\ker G$ *is dense in* $\mathcal{X}$*,*

and if there exists $\beta \in \mathbb{C}$ such that

(iii) $\beta I - J$ *restricted to* $\ker G$ *is onto;*
(iv) $\ker(\beta I - J) \cap \ker G = \{0\}$.

$\mathcal{Z}$ is called the solution space, $\mathcal{X}$ is the state space and $\mathcal{U}$ is the input space.

The following Proposition 4 gathers well-known results. Proofs can be found in [49, Chapter 10] and the references therein.

Proposition 1. *Let (J, G) be a boundary control system on $(\mathcal{Z}, \mathcal{X}, \mathcal{U})$.*

Denote $\mathcal{X}_1 := \ker G$, $A := J|_{\mathcal{X}_1}$ and $\mathcal{X}_{-1}$ as the completion of $\mathcal{X}$ endowed with the norm $\left\|(\beta I - A)^{-1} \cdot\right\|_{\mathcal{X}}$ for some fixed $\beta \in \rho(A)$. Then, the following holds:

1. *$\mathcal{X}_1$ is a Hilbert space endowed with the graph norm of A, as well as a continuously embedded closed subspace of $\mathcal{Z}$ (generally not densely embedded);*

2. *$A \in \mathcal{L}(\mathcal{X}_1, \mathcal{X})$ and can be continuously extended to an operator $A|_{\mathcal{X}}$ in $\mathcal{L}(\mathcal{X}, \mathcal{X}_{-1})$. Furthermore, if A is skew-adjoint on $\mathcal{X}$, then $A|_{\mathcal{X}}$ is skew-adjoint on $\mathcal{X}_{-1}$;*
3. *for $\beta \in \mathbb{C}$ as in Definition 4, $\beta \in \rho(A)$, i.e., in the resolvent set of A, and $(\beta I - A)^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_1)$, $(\beta I - A|_{\mathcal{X}})^{-1} \in \mathcal{L}(\mathcal{X}_{-1}, \mathcal{X})$.*
   Furthermore, the graph norm of A on $\mathcal{X}_1$ is equivalent to the norm $\|(\beta I - A) \cdot \|_{\mathcal{X}}$;
4. *there exists a unique control operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$ such that*

$$J = A|_{\mathcal{X}} + BG, \qquad G(\beta I - A|_{\mathcal{X}})^{-1}B = I_{\mathcal{U}};$$

   furthermore, the operator $\begin{bmatrix} I_{\mathcal{Z}} \\ G \end{bmatrix}$ is a bounded bijection between $\mathcal{Z}$ and $\left\{ \begin{pmatrix} z \\ u \end{pmatrix} \in \mathcal{X} \times \mathcal{U} \mid A|_{\mathcal{X}} z + Bu \in \mathcal{X} \right\}$;
5. *$\mathcal{Z} = (\beta I - A|_{\mathcal{X}})^{-1} (\mathcal{X} + B\mathcal{U}) = \mathcal{X}_1 + (\beta I - A|_{\mathcal{X}})^{-1} B\mathcal{U}$ and B is strictly unbounded, meaning that $\mathcal{X} \cap B\mathcal{U} = \{0\}$, and is bounded from below. In particular, for all $z \in \mathcal{Z}$, there exists a unique $z_0 \in \mathcal{X}_1$ and a unique $u \in \mathcal{U}$ such that $z = z_0 + (\beta I - A|_{\mathcal{X}})^{-1} Bu$.*

B. Proof of Theorem 3

Let us start by showing that (J, G) is a boundary control system on $\left(\mathcal{Z}^1 \times \mathcal{Z}^2, \mathcal{X}^1 \times \mathcal{X}^2, \mathcal{U}^1 \times \mathcal{U}^2\right)$. The four points of Definition 4 have to be checked.

Point (i)**:** Since $\gamma^i \mathcal{Z}^i = \mathcal{U}^i$, $i = 1, 2$, by assumption $(A1)$, $G(\mathcal{Z}^1 \times \mathcal{Z}^2) = \mathcal{U}^1 \times \mathcal{U}^2 =: \mathcal{U}$, i.e., point (i) of Definition 4 holds.

Point (ii)**:** Since $\mathcal{X}_1 := \ker G = \left\{ \begin{pmatrix} e^1 \\ e^2 \end{pmatrix} \in \mathcal{Z} \mid \gamma^1 e^1 = 0, \gamma^2 e^2 = 0 \right\} = \ker \gamma^1 \times \ker \gamma^2 =: \mathcal{X}_1^1 \times \mathcal{X}_1^2$, by assumption $(A2)$, $\mathcal{X}_1$ is then dense in $\mathcal{X}$ and point (ii) of Definition 4 is satisfied.

Point (iii)**:** By assumptions $(A1)$, $(A2)$ and $(A3)$, Theorem 2 applies and A is skew-adjoint on $\mathcal{X}$; so, in particular, $(\beta I - A)$ is onto for all $\beta \in \mathbb{C}$, $\Re\mathrm{e}\beta \neq 0$; the point (iii) of Definition 4 holds.

Point (iv)**:** Let $J := \begin{bmatrix} 0 & -K \\ L & 0 \end{bmatrix}$ and $e \in \ker(I - J) \cap \mathcal{X}_1$. Then, we have the following:

$$e = Ae \in \mathcal{X}_1.$$

Applying $A^\star = -A$, by Theorem 2, one gets

$$-Ae = A^\star Ae \in \mathcal{X},$$

from which it is deduced that

$$e = -A^\star Ae \in \mathcal{X}.$$

Multiplying both sides by e in $\mathcal{X}$, we obtain $\|e\|^2_{\mathcal{X}_1} = 0$. Then, $\ker(I - J) \cap \mathcal{X}_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ and point (iv) of Definition 4 holds.

This shows that (J, G) is indeed a boundary control system on $(\mathcal{Z}^1 \times \mathcal{Z}^2, \mathcal{X}^1 \times \mathcal{X}^2, \mathcal{U}^1 \times \mathcal{U}^2)$. As a first consequence, the control operator B is uniquely determined, as claimed in Proposition 4, point 4.

Stokes-Dirac structure: Starting from (2.8) with the definition of $C := \begin{bmatrix} 0 & \beta^2 \\ \beta^1 & 0 \end{bmatrix}$, one has the following, for all $z := \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in \mathcal{Z}$ and all $x := \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathcal{Z}$:

$$\begin{aligned} (Jz, x)_{\mathcal{X}} + (z, Jx)_{\mathcal{X}} &= \left(-Kz^2, x^1\right)_{\mathcal{X}^1} + \left(Lz^1, x^2\right)_{\mathcal{X}^2} + \left(z^1, -Kx^2\right)_{\mathcal{X}^1} + \left(z^2, Lx^1\right)_{\mathcal{X}^2}, \\ &= \left\langle \gamma^1 z^1, \beta^2 x^2 \right\rangle_{\mathcal{U}^1, (\mathcal{U}^1)'} + \left\langle \beta^1 z^1, \gamma^2 x^2 \right\rangle_{(\mathcal{U}^2)', \mathcal{U}^2} \\ &\quad + \left\langle \beta^2 z^2, \gamma^1 x^1 \right\rangle_{(\mathcal{U}^1)', \mathcal{U}^1} + \left\langle \gamma^2 z^2, \beta^1 x^1 \right\rangle_{\mathcal{U}^2, (\mathcal{U}^2)'}, \\ &= \langle Gz, Cx \rangle_{\mathcal{U}, \mathcal{U}'} + \langle Cz, Gx \rangle_{\mathcal{U}', \mathcal{U}}. \end{aligned}$$

From Proposition 4, point 4, $J = \begin{bmatrix} A|_{\mathcal{X}} & B \end{bmatrix} \begin{bmatrix} I_{\mathcal{Z}} \\ G \end{bmatrix}$, and, thus, with the definitions of $\mathcal{F}$ and $\mathcal{E}$, one has the following, for all $\begin{pmatrix} z \\ u \end{pmatrix} \in \mathcal{E}$ and all $\begin{pmatrix} x \\ v \end{pmatrix} \in \mathcal{E}$:

$$\begin{aligned} \left\langle \begin{bmatrix} A|_{\mathcal{X}} & B \\ -C & 0 \end{bmatrix} \begin{pmatrix} z \\ u \end{pmatrix}, \begin{pmatrix} x \\ v \end{pmatrix} \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \begin{pmatrix} z \\ u \end{pmatrix}, \begin{bmatrix} A|_{\mathcal{X}} & B \\ -C & 0 \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \right\rangle_{\mathcal{F},\mathcal{E}} &= (Jz, x)_{\mathcal{X}} + (z, Jx)_{\mathcal{X}} - \langle Cz, v \rangle_{\mathcal{U}', \mathcal{U}} - \langle u, Cx \rangle_{\mathcal{U}, \mathcal{U}'}, \\ &= \langle Gz, Cx \rangle_{\mathcal{U}, \mathcal{U}'} + \langle Cz, Gx \rangle_{\mathcal{U}', \mathcal{U}} \\ &\quad - \langle Cz, Gx \rangle_{\mathcal{U}', \mathcal{U}} - \langle Gz, Cx \rangle_{\mathcal{U}, \mathcal{U}'}, \\ &= 0. \end{aligned}$$

This yields that $\mathcal{J} := \begin{bmatrix} A|_{\mathcal{X}} & B \\ -C & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ indeed satisfies (2.5).

Applying Theorem 1 shows that the graph of $\mathcal{J}$ defined as above is a Stokes-Dirac structure on $\mathcal{B} = \mathcal{F} \times \mathcal{E}$.

Form of $\mathcal{J}$: Now, it remains to be proven that $\mathcal{J} = \begin{bmatrix} A|_{\mathcal{X}} & B \\ -C & 0 \end{bmatrix}$ can be written as in (2.11) by showing that indeed

$$B = \begin{bmatrix} 0 & B^2 \\ B^1 & 0 \end{bmatrix},$$

with $B^1 \in \mathcal{L}(\mathcal{U}^1, \mathcal{X}^2_{-1})$, $B^2 \in \mathcal{L}(\mathcal{U}^2, \mathcal{X}^1_{-1})$, where we recall that $\mathcal{X}^i_{-1}$ is the projection of $\mathcal{X}_{-1}$ on the i-th component for $i = 1, 2$.

The form of B entirely relies on its construction, as given in the proof of [49, Proposition 10.1.2] $B = (J - A)H$, where $H \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ is a bounded right inverse of G (which exists since G is onto).

Since $G = \begin{bmatrix} \gamma^1 & 0 \\ 0 & \gamma^2 \end{bmatrix}$, $H = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}$, where $H^i \in \mathcal{L}(\mathcal{U}^i, \mathcal{Z}^i)$ is a bounded right inverse of γ^i for $i = 1, 2$. By construction with the operators K and L and the assumption of density of $\mathcal{X}_1$ in $\mathcal{X}$, $J - A|_{\mathcal{X}} = BG$ is of the form $\begin{bmatrix} 0 & S^2 \\ S^1 & 0 \end{bmatrix}$, which yields that $B = \begin{bmatrix} 0 & S^2 H^2 \\ S^1 H^1 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{U}^1 \times \mathcal{U}^2, \mathcal{X}^1_{-1} \times \mathcal{X}^2_{-1})$. Hence, $B^1 = S^1 H^1$ is related to γ^1, and $B^2 = S^2 H^2$ is related to γ^2. This concludes the proof.